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Evolutionary dynamics of the ultimatum game

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Abstract

In the ultimatum game two players are offered the chance to share a prize (e.g. money). All that is needed is for them to agree on the division. One player makes an offer which, if accepted, is the split but, if rejected, both payers get nothing. The strategies predicted by standard game theory is to offer very little and accept anything. Humans, however, in both roles, usually prefer a fairer split. In this paper we study the ultimatum game within the framework of evolutionary game theory to see if strategies closer to those observed in reality can emerge.

Contents

1 Preliminary

1.1 The ultimatum game

The ultimatum game is a 2-player game in which a prize e.g. (and most commonly in experiments) a sum of money is shared if a split can be agreed. The first player (the proposer) proposes a split. The second player (the responder) can either accept or reject the offer. If the offer is accepted the players keep those splits whereas if the offer is rejected both players get nothing.

The game was introduced by G*ü*th et al [1] in 1982 and has since, along with the prisoner's dilemma, become a "prime showcase of apparently irrational behaviour" [3]. According to game theory the 'rational' way for the responder to play is to accept any offer (no matter how small) as the responder has simply an option between a) the offer i.e. something and b) nothing. The rational proposer who expects the responder to behave rationally should therefore offer the responder a low share.

It has, however, been demonstrated that people do not play the game rationally. Most proposers offer a fairer share. Experimental results show that around 60-80% offer fractions between 0.4 and 0.5 and only 3% offer less than 0.2 . This seems like a reasonable strategy given that around 50% of responders reject splits offering less than one-third of the total. See [5] [6] [7] [9] [8] [10] [12] [13] [11].

Experimental data has also demonstrated that increasing the stake size has little effect on player's strategies. See e.g. [14] where the experiment was done by survey in Australia but even more convincing see [11] where the game was played in Indonesia with stake sizes up to three times the average monthly expenditure.

The extensive studies have shown similar behaviour across a range of different circumstance and cultural environments (e.g. Jerusalem, Llubljana & Pittsburgh [7] and Yogyakarta Indonesia [11]). Whilst this is evidence that culture does not have a significant influence on ultimatum game behaviour this has been questioned, more recently, by an experimental study in a non-industrialised society, specifically amongst the Machiguenga people in the Peruvian Amazon which suggests that cultural differences may greatly influence economic behaviour [16]. Amongst the Machiguenga the mean offer was only 0.26 and responders almost always accepted offers less than 0.2.

In a biological context, the ultimatum game could describe reward sharing in advance of a task between two individuals where the dominant one issues a 'take it ir leave it' ultimatum. The game could also reflect instances of resource sharing and allocation where if agreement is not reached the opportunity may be lost (e.g. the food might run away). "Hence, while the experimental situation of an isolated, anonymous ultimatum game is somewhat artificial, it is very likely that situations similar to it have shaped the fairness instinct of animals and humans for millions of years" [4]. It has been shown e.g. that inequity aversion is displayed by brown capuchin monkeys (*Cebus Capella*) who seemed to respond negatively to unequal reward distribution in exchanges with a human experimenter [15].

Whether the relevant behaviours are acquired through biological reproduction or by social learning and cultural imitation a useful framework to study evolving strategies is evolutionary dynamics and evolutionary game theory. The evolutionary dynamics of the ultimatum game will be reviewed in this paper using both evolutionary/genetic algorithms and replicator equations. A broad aim is to investigate whether so-called 'fair' strategies can evolve in populations made up of individuals looking to maximise their rewards.

2 Evolutionary Algorithms

Evolutionary algorithms (EAs) are optimisation algorithms that apply mechanisms of biological evolution (in this paper reproduction, selection and mutation) to find solutions. EAs are applied in many diverse fields and seem an obvious application in this paper as the process of change is most likely biological in which case the mechanisms would be as in evolutionary biology but if the process of change of behaviour is social the EA would still be a useful framework and the mechanisms would represent imitation, learning and variation. We will apply EAs to populations in a number of environments to study how behaviour might evolve.

2.1 Random encounters in a well-mixed population

For simplicity suppose we take the sum to be divided as unity. Strategies of individuals in the population are given by two parameters p and q both on the interval $[0, 1]$.
p denotes the amount offered if proposing.
q denotes the minimum accepted if responding.

Suppose that in an interaction between a player using strategy $S_1 = (p_1, q_1)$ and a player using strategy $S_2 = (p_2, q_2)$ each has an equal chance of playing either role. The expected value of the pay-off for the S_1 player against the S_2 player, $E(S_1, S_2)$ is

$$E(S1, S2) = \frac{1}{2} \times \begin{cases} 1 - p_1 + p_2 & \text{if } p_1 \geq q_2 \text{ and } p_2 \geq q_1 \\ 1 - p_1 & \text{if } p_1 \geq q_2 \text{ and } p_2 < q_1 \\ p_2 & \text{if } p_1 < q_2 \text{ and } p_2 \geq q_1 \\ 0 & \text{if } p_1 < q_2 \text{ and } p_2 < q_1 \end{cases} \tag{2.1}$$

Consider the following environment for studying the evolutionary dynamics in well-mixed population:
• Population - There is a population of N individuals.
• Interactions - In each generation every player interacts with every other player in the role of both proposer and responder.
• Fitness - The fitness of an individual is equal to the sum of the payoffs accumulated from each interaction.
• Selection - Players leave offspring in proportion to their fitness i.e. the probability of each child taking the strategy of parent i is given by $\frac{F_i}{\sum_{j=1}^{N} F_j}$ where F_i denotes fitness of the i^{th} individual. This is known as roulette wheel selection.
• Mutation - Offspring inherit the parent's strategy subject to a small mutation. If p_i and q_i are the parent's strategies, the child's strategies will be on the interval $[p_i - \frac{\epsilon}{2}, p_i + \frac{\epsilon}{2}]$ and $[q_i - \frac{\epsilon}{2}, q_i + \frac{\epsilon}{2}]$ with ϵ a small mutation parameter.

Figure 2.1 shows how the evolution of (initially random) strategies of one simulation of a population of 100 individuals with $\epsilon = 0.01$ evolve towards the rational. Table 2.1 shows the results of simulations run with various mutation errors, ϵ. With small mutation errors the population favours the rational however larger mutation errors result in non-zero acceptance levels which creates selective pressure on the proposer away from the rational. These results are consistent with the study by Page *et al* in [4].

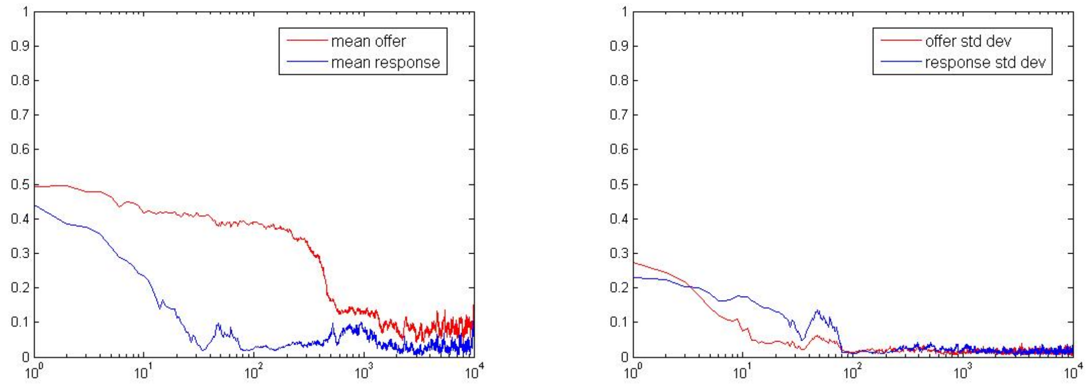

Figure 2.1: The left figure shows the time evolution of the average offer and acceptance level in a simulation of the non-spatial, well-mixed ultimatum game. Initially the 100 players each have random offer and acceptance levels. Everyone plays everyone else (both as proposer and responder) and the number of offspring of each individual is proportional to his total pay-off. The mutation error, ϵ, is 0.01. The time scale is logarithmic to illustrate the long term (though noisy) convergence. The right figure shows the standard deviation of the population strategies.

ϵ	$\bar{p}$	$\bar{q}$
0.001	0.0807 ±0.0224	0.0637 ±0.0245
0.002	0.0926 ±0.0382	0.0704 ±0.0386
0.01	0.1065 ±0.0296	0.0469 ±0.0275
0.02	0.1452 ±0.0265	0.0571 ±0.0222
0.1	0.2683 ±0.0336	0.0989 ±0.0244
0.2	0.3224 ±0.0403	0.1285 ±0.0243

Table 2.1: Summary of results in the non-spatial, well mixed ultimatum game. The table shows the mean offer and acceptance level in a population of 100 individuals with various mutation errors, ϵ. All individuals have random initial strategies. The values shown are averages over time, sampled at 10^3 generation intervals between 10^4 generations and 10^5 generations. The standard deviations are of the sample and are a measure of how stable the population strategies have become. The results are consistent with [4]

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2.2 The spatial ultimatum game in one dimension

What if the population does not interact randomly and completely? If there is some social structure to the population due to spatial arrangement how will the strategies evolve?

Consider the following spatial environment:

• Population - There is a population of N individuals arranged on a one-dimensional ring (or annulus).
• Interactions - In each generation every player interacts with his k nearest neighbours.
• Fitness - The fitness of an individual is equal to the sum of the payoffs accumulated from these two interactions.
• Selection - Players leave offspring in proportion to their relative fitness in their neighbourhood i.e. the probability that a child at a specific site takes the strategy of a parent in the neighbourhood of that site is equal to the parent's fitness divided by the total fitness of the three players in that neighbourhood.
• Mutation - as before.

When competing for offspring in a well-mixed environment an individual's fitness is compared to the entire population. When competing for offspring in a neighbourhood a player can affect his neighbour's payoff and beating your neighbours is what is important for survival. When do mutant clusters spread? Page *et al* in [4] showed conditions for a mutant with strategy $S_2 = (p_2, q_2)$ to be likely to invade strategy $S_1 = (p_1, q_1)$ where $q_1 \leq p_1 < q_2 \leq p_2$. These are shown in figure 2.2. They demonstrate why with neighbourhood $k = 2$ we can expect to see strategies evolving that tend towards an even split. These spatial dynamics are demonstrated in figure 2.3. Table 2.2 shows the results of simulations done for a different neighbourhood sizes.

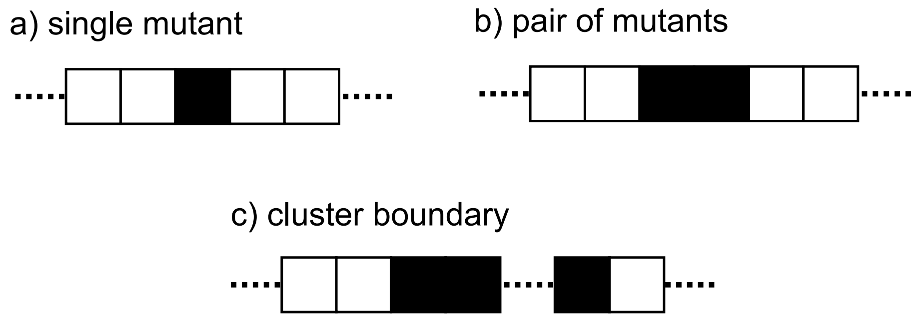
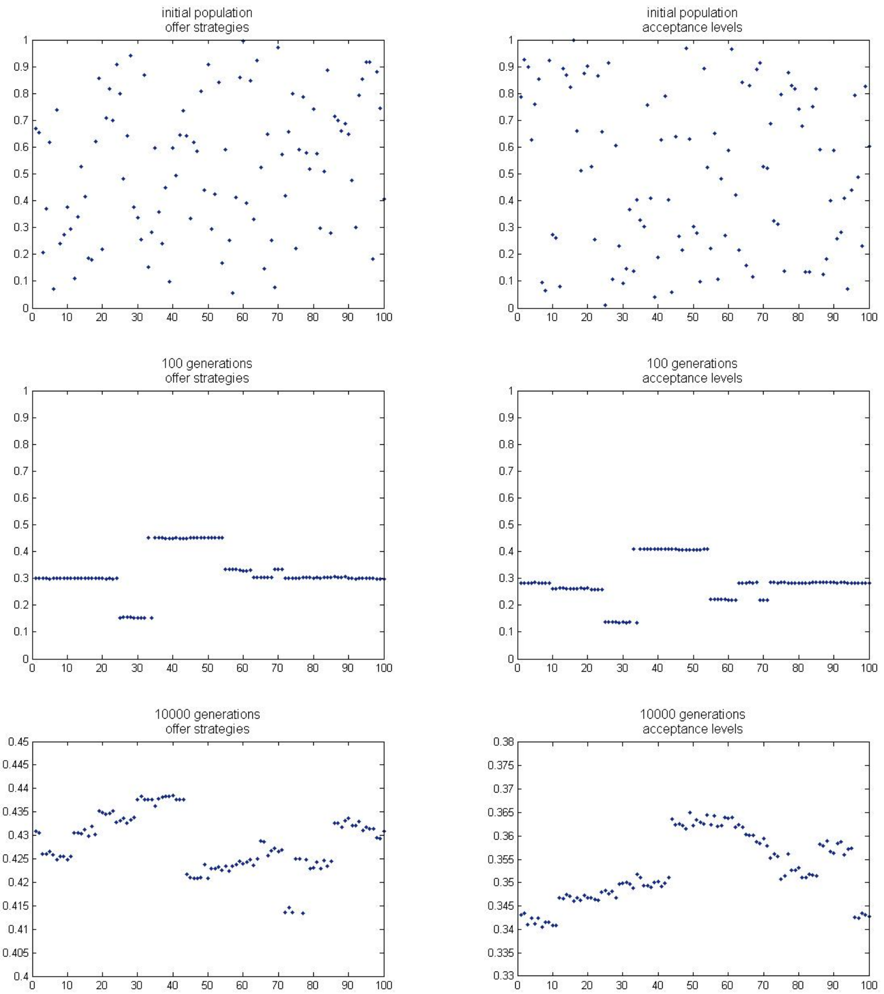


Figure 2.2: Recreated from [4]. In a population of $S_1 = (p_1, q_1)$ strategists, a cluster of mutants playing $S_2 = (p_2, q_2)$ where $q_1 \leq p_1 < q_2 \leq p_2$ are likely to propagate if *a*) $p_2 \leq 0.39$, *b*) $p_2 \leq 0.43$, and *c*) $p_2 \leq 0.5$

neighbours	$\bar{p}$	$\bar{q}$
2	0.4170 ±0.0056	0.3905 ±0.0616
4	0.2948 ±.0865	0.2742 ±0.0903
8	0.2372 ±0.0287	0.2208 ±0.0903

Table 2.2: Summary of results in a one-dimensional ring of different neighbourhood sizes. The table shows the mean offer and acceptance level in a population of 100 individuals with different neighbourhood sizes. All individuals have random initial strategies. The values shown are averages over time, sampled at 10^3 generation intervals between 10^4 generations and 10^5 generations. The standard deviations are of the sample and are a measure of how stable the population strategies have become. $k = 2$ is consistent with [4] but $k = 4$ and $k = 8$ are lower. A linear extrapolation from [4] suggests we should expect proposal strategies at about 0.39 for $k = 4$ and 0.28 for $k = 8$. In that paper generations were sampled between 10^5 and 10^6 generations so the populations in the simulations tabled here may still be climbing. Given a little more time it would be useful to run the simulations again and for longer to investigate.

Figure 2.3: The figures show snapshots of strategies evolving in a one-dimensional ring. Initially the strategies are arranged randomly. After 100 generations the population has arranged itself into a number of separate clusters. Much later one of the clusters has dominated and clustering occurs on a small scale as the players evolve towards fair strategies. The mutation error ϵ is 0.001.



2.3 The spatial ultimatum game in two dimensions

Now let us look at the evolution of strategies in a two-dimensional spatial environment. Consider an environment with evolutionary dynamics as in the one-dimensional case but with the population arranged on a two-dimensional square lattice. Each player interacts with his neighbours above, below, to the left and to the right. This is known as a Von-Neumann neighbourhood. Also the lattice folds around to form a torus so that there are no boundaries and each player has the same size neighbourhood.

When do mutant clusters spread? Page *et al* in [4] examined a 3×3 cluster and showed the condition for a mutant with strategy $S_2 = (p_2, q_2)$ to be likely to invade strategy $S_1 = (p_1, q_1)$ where $q_1 \leq p_1 < q_2 \leq p_2$ is $p_2 \leq 0.342$. This cluster is shown in figure 2.4. We can therefore expect clusters to form and grow up to around this level. Figure 2.5 shows snapshots of the evolution of the population strategies in which we can see clusters forming and spreading. Table 2.3 shows the results of simulations done for a different neighbourhood sizes. The results are all consistent with [4] As in the one-dimensional environment strategies away from zero are able to evolve.

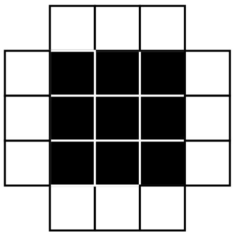
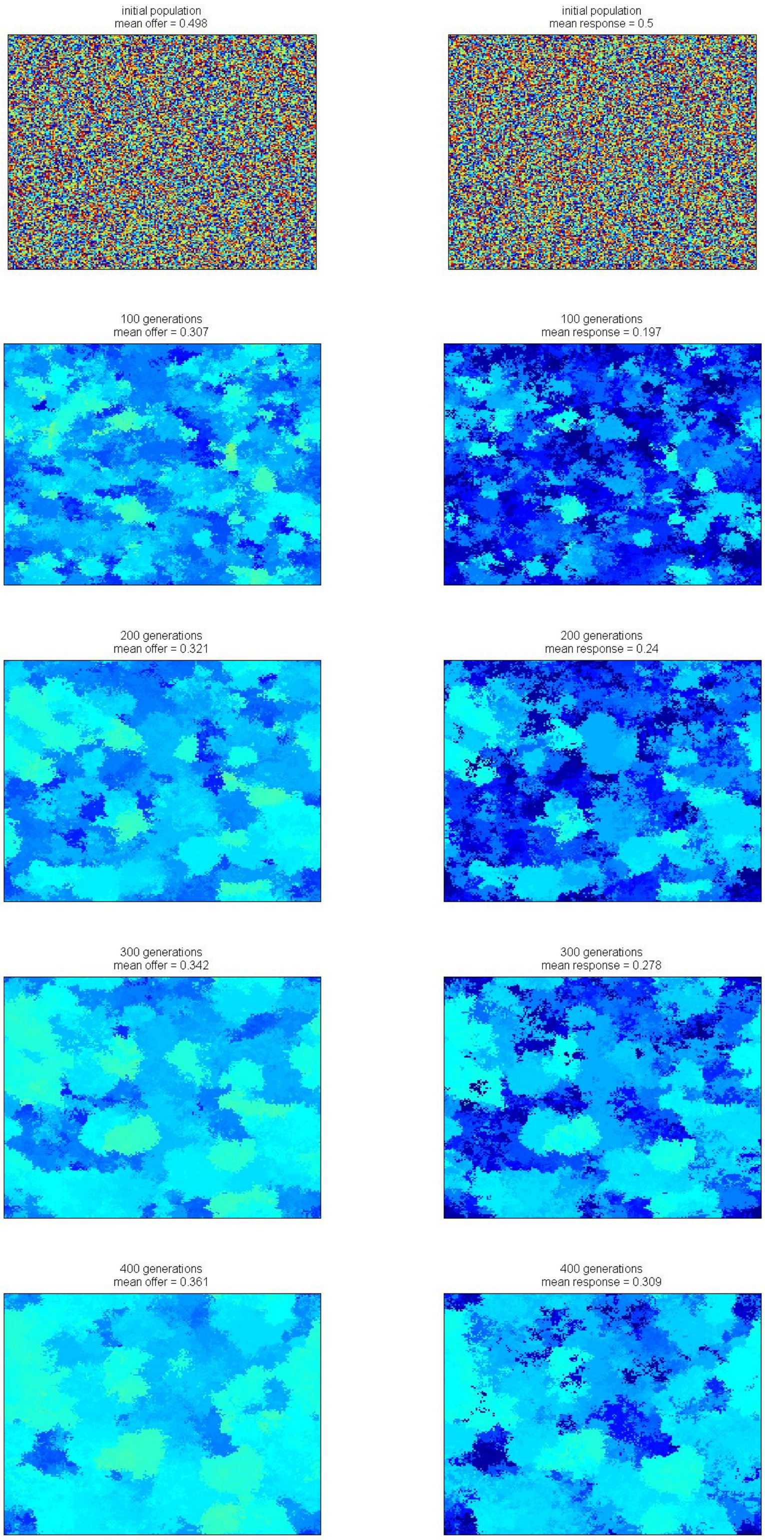


Figure 2.4: Recreated from [4]. In a population of $S_1 = (p_1, q_1)$ strategists, a $3 \times s$ cluster of mutants (black cells) playing $S_2 = (p_2, q_2)$ where $q_1 \leq p_1 < q_2 \leq p_2$ are likely to propagate if $p_2 \leq 0.342$.

grid size	$\bar{p}$	$\bar{q}$
10 x 10	0.2435 ±0.0381	0.2240 ±0.0392
50 x 50	0.3088 ±0.0130	0.2836 ±0.0164
100 x 100	0.3328 ±0.0109	0.3046 ±0.0117

Table 2.3: Summary of results on two-dimensional grids of various sizes. The table shows the mean offer and acceptance level on various grid sizes e.g. the $10x10$ contains 100 individuals. All individuals have random initial strategies. The values shown are averages over time, sampled at 10^3 generation intervals between 10^4 generations and 10^5 generations. The standard deviations are of the sample and are a measure of how stable the population strategies have become. These results are all consistent with [4].

Figure 2.5: The figures show snapshots of strategies evolving on a two-dimensional grid. The colour scale is a heat colour map - the range $[0, 1]$ is charted as dark blue through to red. Importantly for the figures below lighter means higher. Initially all the strategies are random. Early on selection favours low offer and acceptance levels. Once clusters establish themselves they are able to spread and to evolve higher strategies with the acceptance levels lagging a little behind the offers.



2.4 The ultimatum game on a graph

What if the spatial distribution of the social structure is less ordered?

We have seen that a spatial distribution where individuals are constrained to interact with, and imitate, only their neighbours can lead to fairer strategies. We have also seen that the topology of the environment makes a difference - the $k = 4$ (four neighbour) case on the one-dimensional ring resulted in fairer strategies than $k = 4$ on the two-dimensional lattice. Is the evolution of fairer strategies driven purely by the spatial distribution or is it due, in part, to the restriction of interactions to a small number of neighbours? We can examine this question by studying the evolutionary dynamics of the game on a random regular graph where each vertex represents an individual and edges denote interactions.

The regular graph has the following properties:
• Loops are not allowed i.e. an individual does not play against himself.
• Multiple edges are not allowed i.e. individuals play each other at most once per generation.
• Each vertex has the same number of edges i.e. individuals play the same number of games.
The graph can be represented by an adjacency matrix $A = [a_{ij}]$ where:
if $a_{ij} = 1$ then $i \leftrightarrow j$ i.e. i and j interact, and
if $a_{ij} = 0$ then i and j do not interact.

The algorithm for generating the random graph graph works as follows: [18] [19]
• Begin with a regular graph with k edges connected to the nearest neighbours i.e. a ring with k neighbours.
• Randomly select a pair of edges $A \leftrightarrow B$ and $C \leftrightarrow D$.
• Rewire so that $A \leftrightarrow D$ and $C \leftrightarrow B$.
• If one or both of these edges already exist the step is aborted and a new pair chosen.
• Repeat rewiring a number of times. Here we have used $4N$ times.

2.4 shows the results of simulations of a population of 100 individuals arranged randomly on a regular graph. Evolutionary dynamics are as in the one- and two-dimensional ordered cases. The less ordered structure makes it difficult for clusters to form - your neighbours neighbour may be far away from you. This reduces the impact of kin selection and makes it more difficult for fair strategies to evolve.

neighbours	$\bar{p}$	$\bar{q}$
2	0.4170 ±0.0056	0.3905 ±0.0616
4	0.1026 ±0.0398	0.0841 ±0.0378
8	0.1075 ±0.0174	0.0886 ±0.0163

Table 2.4: Summary of results of the ultimatum game on a graph of different neighbourhood sizes. The table shows the mean offer and acceptance level in a population of 100 individuals for $k = 2, 4$ & 8. All individuals have random initial strategies. The values shown are averages over time, sampled at 10^3 generation intervals between 10^4 generations and 10^5 generations. The standard deviations are of the sample and are a measure of how stable the population strategies have become. $k = 2$ is taken from 2.2 as any connected regular graph of degree 2 is planar and equivalent to a ring. The $k = 4$ and $k = 8$ case have significantly lower strategies than the $k = 2$ case reflecting the importance of kin selection. The results are consistent with [17] where the effect of the topology of the spatial environment was studied. Note that the $k = 4$ case is slightly lower than the $k = 8$ case. This may again be due to the population requiring more generations to settle down - the higher standard deviation suggests this is the case, however it would be useful to run the simulations again and for longer to investigate.

2.5 Cost of interaction and refusal to play

What would happen if individuals were allowed to be more selective about who they interacted with? If we imagine that each interaction involves a small cost (this could be a set-up cost or simply a time opportunity cost) individuals would prefer not to incur this cost if they are not likely to reach agreement.

Consider the following evolutionary dynamics:
• Population - There is a population of N individuals.
• Interactions - From the population of N individuals a proposer and responde are selected randomly. If they do not reach agreement they will black ball each other with probability σ. If they have previously black balled each other they will not play and thereby not incur the interaction cost, c. Per generation there are $m \times N$ interactions - enough for repeated interactions between individuals.
• Fitness - We now have the possibility of negative payoffs that need to be taken into account. We could use a linear map from the full range of possible payoffs to positive i.e. $[z_{min}, z_{max}] \to [0, 1]$ where
z_{min} is the payoff of the individual selected every time and always incurring a cost, and
z_{min} is the payoff of the individual selected every time and always scoring 1.
The possibility of these extremes occurring are obviously remote so to avoid slow convergence (tight grouping of fitness scores will mean weak selection) we will calculate a more realistic high, low range as follows:
the probability of an individual being selected in in each of the mN interactions is $\frac{2}{N}$.
This forms the binomial distribution

$$\mathbb{P}(X = k) = \binom{mN}{k} \left(\frac{2}{N}\right)^k \left(1 - \frac{2}{N}\right)^{mN-k}, \qquad k = 0, 1, ..., nM$$

where X is the number of interactions an individual is selected for in a generation. We can use the cumulative distribution function to find with say a 90% confidence that an individual will not play more than $\tilde{X}_{max}$ games. The adjusted high, low payoff range is

then:
$\tilde{z}_{min} = \tilde{X}_{max} \times (-c)$
$\tilde{z}_{max} = \tilde{X}_{max}$
• Selection - Selection is as in the well-mixed environment i.e. individuals compete for offspring with the entire population. Any individuals with negative fitness do not reproduce.
• Mutation - as before

The social structure introduced here was not able to encourage 'fair' behaviour. When a 'fairer' player blackballs an uncooperative one the benefit derived is that in a future interaction he will not incur the setup cost - without the setup cost they would simply both again score zero in a future interaction. Without the setup cost the foregone payoff if agreement is not reached is $1-p_1$ if proposing and p_1 if responding. With the setup cost the foregone payoff is $1-p_1-c$ if proposing and p_1-c if responding. So for blackballing, the 'fairer' player is now c better off relative to the entire population. However the same is true of the uncooperative player. For strategies towards even shares to evolve seemed to require costs that are unrealistic in the context of why they were proposed. Figure 2.6 shows two examples of time evolutions of strategies. After a few hundred generations the populations tended to become very homogenous after which there is very little blackballing and the evolutionary dynamics are similar to that of a population without these new drivers.

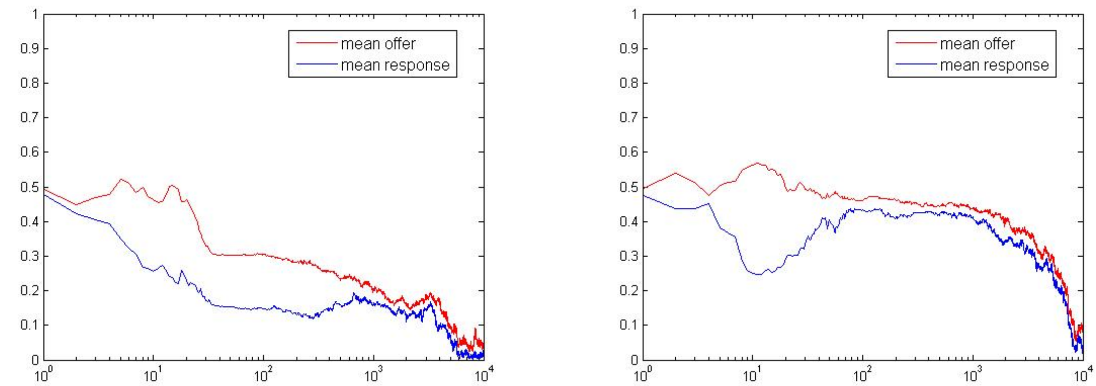


Figure 2.6: The figure shows two examples of the time evolution of strategies with cost, $c = 4$. After a few hundred generations the populations become highly homogenised and there are few blackballs. Selective pressure as in the well-mixed case drives strategies down to close to zero.

Despite the results this could still be an interesting structure to study. A possible change might be where other players ar able to see that a player is blackballed and this label on the uncooperative player may cause others to refuse an interaction.

3 Replicator Dynamics

3.1 Motivation for the standard replicator equations

The replicator equation is one of the fundamental equations of evolutionary dynamics. It describes population dynamics where successful strategies spread, either by cultural imitation or biological reproduction [2] in an infinitely large, well-mixed population. They were introduced by Taylor & Jonker in 1978 [20]. Below is a brief derivation.

In a large population each individual plays one of the strategies S_1,...,S_n.
The frequencies of the n strategies are denoted by $x_1, ..., x_n$.
$A = (a_{ij})_{i,j=1}^{n}$ is the game payoff matrix i.e. a_{ij} is the payoff for the pairwise encounter of S_i and S_j.

The fitness of $S_i = C + (Ax)_i$ where $C > 0$ is a basic fitness of the population.
Each S_i individual produces S_i offspring (cloning) in proportion to their frequency and fitness i.e. S_i offspring $\sim x_i(C + (A\mathbf{x})_i)$
Total offspring $\sim \sum\limits_{i=1} x_i(C + (A\mathbf{x})_i) = C + A\mathbf{x}$

The frequencies in the next generation are thus:

$$x_i' = \frac{x_i(C + (A\mathbf{x})_i)}{C + \mathbf{x} \cdot A\mathbf{x}}, \text{ and so}$$

$$x_i' - x_i = \frac{x_i(C + (A\mathbf{x})_i) - x_i(C + \mathbf{x} \cdot A\mathbf{x})}{C + \mathbf{x} \cdot A\mathbf{x}} = \frac{x_i((A\mathbf{x})_i - \mathbf{x} \cdot A\mathbf{x})}{C + \mathbf{x} \cdot A\mathbf{x}}$$

Suppose C is large in comparison to A i.e.

$$x_i' - x_i \approx x_i((A\mathbf{x})_i - \mathbf{x} \cdot A\mathbf{x})\frac{1}{C}$$

Now choose the time scale so that the time between generations $\Delta t = \frac{1}{C}$ i.e.

$$\Delta x_i = x_i((A\mathbf{x})_i - \mathbf{x} \cdot A\mathbf{x})\Delta t$$

If $C \to \infty$, then $\Delta t \to 0$ and

$$\frac{d}{dt}x_i = x_i[(A\mathbf{x})_i - \mathbf{x} \cdot A\mathbf{x}], \qquad i = 1,...,n \tag{3.1}$$

The replicator dynamics describe pure selection dynamics (ignoring stochasticity effects) i.e. mutation is not considered. the dynamics are similar to the roulette wheel selection used in this paper. In the replicator equations the change in frequency of a strategy is equal to the product of it's current frequency and relative fitness whereas in roulette wheel selection the probability of selection is equal to the product of it's current frequency and relative fitness.

We cannot apply the replicator equation to the full ultimatum game The replicator equation describes the dynamics of a game with a fixed number of discrete strategies. We can , however, study the dynamics of a reduced form of the ultimatum game (a so-called minigame) that captures aspects of the full game.

3.2 The mini ultimatum game

The replicator dynamics are used to examine the dynamics of games with discrete strategies $1, ..., n$. This is not the case with the ultimatum game but we can study a reduced form of the game (a so-called minigame) that captures aspects of the full ultimatum game. In the mini ultimatum game there are only two possible offers h and l (high and low). Individuals are faced with a choice of four strategies:
$S_1(l, l)$ - offer low, accept low ('rational')
$S_2(h, h)$ - offer high, accept high ('fair')
$S_3(h, l)$ - offer high, accept low
$S_4(l, h)$ - offer low, accept high

Table 3.2 shows the payoff matrix $A = [a_{ij}]_{i,j=1}^4$ where a_{ij} is the payoff i receives when encountering j.

	$S1$	$S2$	$S3$	$S4$
$S1$	1	h	$1-l+h$	l
$S2$	$1-h$	1	1	$1-h$
$S3$	$1-h+l$	1	1	$1-h+l$
$S4$	$1-l$	h	$1-l+h$	0

Table 3.1: Payoff matrix of the mini ultimatum game

Both S_1 and S_2 are Nash equilibria but S_1 is also a strict Nash equilibrium and therefore an evolutionary stable strategy. It has been shown that if a strategy is evolutionary stable or a strict Nash equilibrium, then the corner point of the simplex corresponding to the population playing this strategy is an asymptotically stable fixed point. We can thus expect to see of strategy S_1.
Now S_1 dominates S_4 (a player looking to maximise his payoff would never choose S_4 over S_1) so we can omit S_4 from the game.
Also it is known that the dynamics are unaffected by altering each column by a constant so we can deduct 1 from each column to arrive at the following simpler payoff matrix.

	$S1$	$S2$	$S3$
$S1$	0	$h-1$	$h-l$
$S2$	$-h$	0	0
$S3$	$l-h$	0	0

Table 3.2: Adjusted payoff matrix

3.3 Replicator equations on the mini ultimatum game

Applying the replicator equation 3.1 above to the the payoff table 3.2 above we get the following system of ODE's.

$$\begin{aligned}
\frac{d}{dt}x_1 &= x_1[(h-l)x_3 + (h-1)x_2 + x_1x_2] \\
\frac{d}{dt}x_2 &= x_2[-hx_1 + x_1x_2] \\
\frac{d}{dt}x_3 &= x_3[(l-h)x_1 + x_1x_2]
\end{aligned} \tag{3.2}$$

As the variables are frequencies and $\Sigma_{i=1}^{3} x_i = 1$ we can use apply the dependency $x_3 = 1 - x_1 - x_2$ to reduce the coupled system of equations of three variables to a system of two variables and instead study the following system of ODE's:

$$\begin{aligned}
\frac{d}{dt}x_1 &= x_1[(l-1)x_2 + (h-l)(1-x_1) + x_1x_2] \\
\frac{d}{dt}x_2 &= x_2[-hx_1 + x_1x_2]
\end{aligned} \tag{3.3}$$

Steady states and stability

The steady states of the system (where $\frac{d}{dt}\mathbf{x} = 0$) are:

• the three pure strategies (e_1, e_2 and e_3)
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ - asymptotically stable (strict equilibrium),
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ - stable non-isolated fixed point,
$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ - unstable

• all points on the $x_1 = 0$ simplex boundary
$\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ - stable for $x_2 > \frac{h-l}{1-l}$

• and the point on the simplex boundary between S_1 and S_2
$\begin{pmatrix} 1-h \\ h \end{pmatrix}$ - unstable

Appendix A.1 shows a more detailed analysis of the stability of each of these steady states.

Phase plane

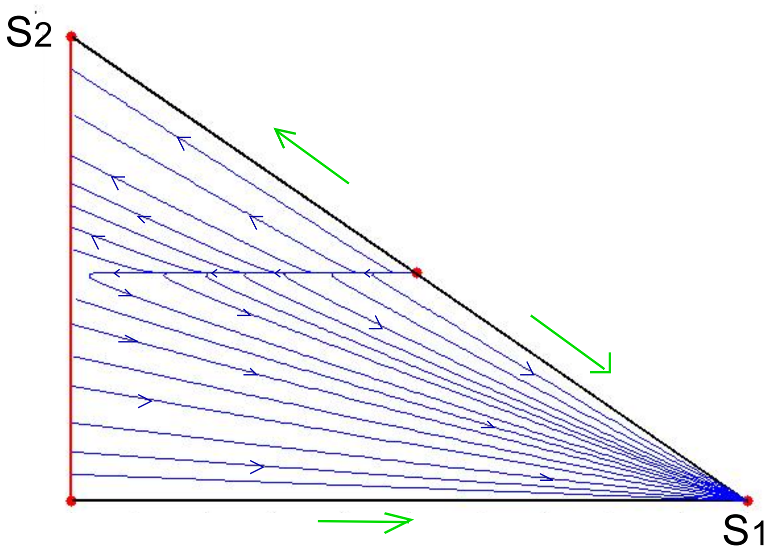


Figure 3.1: Phase plane of the standard replicator dynamics. S_1 ('rational') is an asymptotically stable fixed point. Solutions starting in the region above the basin boundary towards S_2 converge the $x_1 = 0$ boundary. On this stable boundary there is pure drift between S_2 ('fair') and S_3. The $x_1 = 0$ boundary becomes unstable below the basin boundary so, if perturbed, solutions will converge from there to S_1.

3.4 Replicator equations on graphs

As discussed above the replicator equation applies to a well-mixed population where an individual interacts with every other individual (or has an equal likelihood of interacting with every other individual) i.e. population structure is ignored.

Replicator dynamics in structured populations defined by graphs have been studied by Ohtsuki & Nowak [21]. The graphs are as described in 2.4 where individuals occupy the vertices of the graph and edges denote which individuals interact with each other. The replicator equation can be thought of as describing the dynamics on a complete graph where all vertices are connected to each other.

Ohtsuki & Nowak considered three different stochastic processes for their update (selection) rules:
• birth-death - An individual is selected based on his fitness compared to the entire population.
• death-birth - A random individual dies. The neighbours compete for the empty site in proportion to fitness.
• imitation - A random individual is chosen to revise it's strategy and takes either his own or one of his neighbours in proportion to fitness.

Imitation update most closely resembles the selection used elsewhere in this paper so we will focus on that.

Remarkably, their results showed that for games played on random regular graphs (graphs without loops i.e an individual does not play himself, without multiple edges i.e. individuals face off no more than once per generation, and where each vertex has the same number of edges, k i.e. individuals all have the same number of interactions) the derived differential equation for the $k > 2$ case is the replicator equation with an adjusted payoff matrix.

If $A = [a_{ij}]$ is the original $n \times n$ payoff matrix, the transformed payoff matrix $B = [b_{ij}]$ for imitation updating is given by

$$b_{ij} = \frac{(k+3)a_{ii} + 3a_{ij} - 3aji - (k+3)a_{jj}}{(k+3)(k-2)} \tag{3.4}$$

We will examine the dynamics of the mini ultimatum game for the $k = 3$ and $k = 4$ cases.

3.4.1 k=3 graph

Applying the transformation 3.4 to the original payoff matrix 3.2 (with columns each reduced by 1) we get the following

	$S1$	$S2$	$S3$	$S4$
$S1$	0	$2h - \frac{3}{2}$	$2(h-l)$	$2l - \frac{1}{2}$
$S2$	$-2h + \frac{1}{2}$	0	0	$-2h + \frac{3}{2}$
$S3$	$2(l-h)$	0	0	$2(l-h) + 1$
$S4$	$-2l - \frac{1}{2}$	$2h - \frac{5}{2}$	$2(h-l) - 1$	-1

Table 3.3: Adjusted payoff matrix for the $k = 3$ graph.

As before S_1 dominates S_4 and we arrive at the following system of ODE's.

$$\begin{aligned} \frac{d}{dt}x_1 &= x_1[(2l - \frac{3}{2})x_2 + 2(h-l)(1-x_1) + x_1x_2] \\ \frac{d}{dt}x_2 &= x_2[(\frac{1}{2} - 2h)x_1 + x_1x_2] \end{aligned} \tag{3.5}$$

Steady states and stability

The stability of the steady states for the game on the $k = 3$ graph are identical to the well-mixed case (i.e. standard replicator equation) for the three pure strategies but the other two steady states have changed in some way:
For
$\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ - The stability condition is now $x_2 > \frac{2(h-l)}{\frac{3}{2}-l}$

and for the point on the simplex boundary between S_1 and S_2
$\begin{pmatrix} \frac{3}{2} - 2h \\ 2h - \frac{1}{2} \end{pmatrix}$ - the type of equilibrium point has changed to a saddle.

Appendix A.2 shows a more detailed analysis of the stability of each of these steady states.

Phase plane

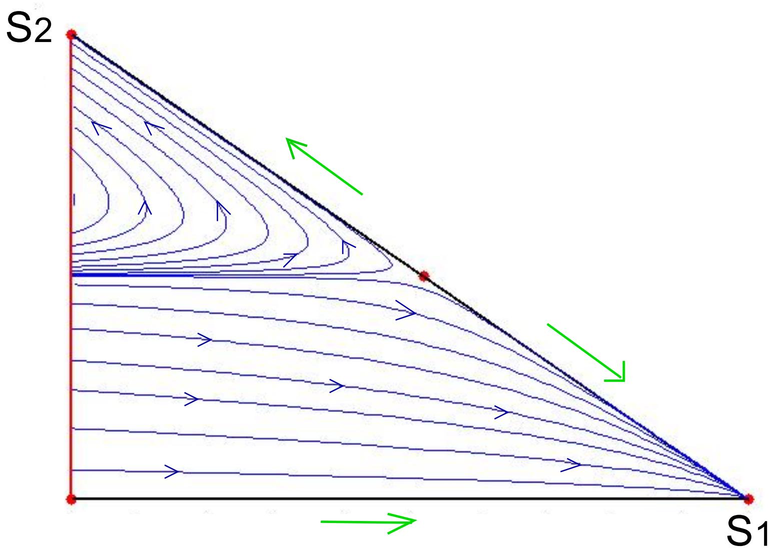


Figure 3.2: Phase plane of the k=3 graph. The fixed point on the boundary between S_1 and S_2 is now a saddle point and the dynamics inside the region close to S_2 have changed. Given drift and perturbations solutions will, however, again converge to S_1.

3.4.2 k=4 graph

	$S1$	$S2$	$S3$	$S4$
$S1$	0	$\frac{10}{7}h - \frac{17}{14}$	$\frac{10}{7}(h-l)$	$\frac{10}{7}l - \frac{5}{7}$
$S2$	$-\frac{10}{7}h + \frac{3}{14}$	0	0	$-\frac{10}{7}h + \frac{5}{7}$
$S3$	$\frac{10}{7}(l-h)$	0	0	$\frac{10}{7}(l-h) + \frac{1}{2}$
$S4$	$-\frac{10}{7}l - \frac{2}{7}$	$\frac{10}{7}h - \frac{12}{7}$	$\frac{10}{7}(h-l) - \frac{1}{2}$	-1

Table 3.4: Adjusted payoff matrix for the $k = 4$ graph

$$\frac{d}{dt}x_1 = x_1[(\frac{10}{7}l - \frac{17}{14})x_2 + \frac{10}{7}(h-l)(1-x_1) + x_1x_2]$$
$$\frac{d}{dt}x_2 = x_2[(\frac{3}{14} - \frac{10}{7}h)x_1 + x_1x_2] \quad (3.6)$$

Steady states and stability

The stability of the steady states of the game on the $k = 4$ graph are very similar to the $k = 3$ graph. The only noteworthy change is the stability condition for the point on the $x_1 = 0$ simplex border which is now $x_2 > \frac{\frac{10}{7}(h-l)}{\frac{17}{14} - \frac{10}{7}}$

Appendix A.3 shows a more detailed analysis of the stability of each of these steady states.

Phase plane

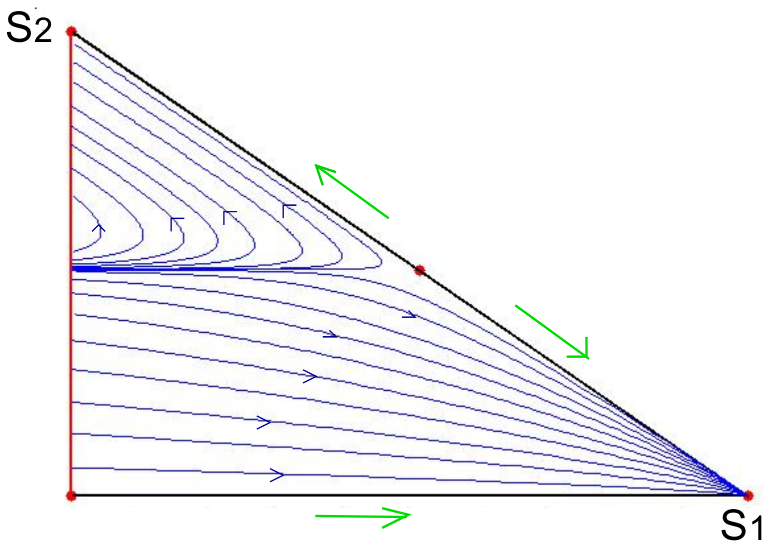


Figure 3.3: Phase plane of the k=4 graph. The dynamics are very similar to the $k = 3$ phase plane 3.2

3.5 Comparison with evolutionary algorithms

What if we applied the evolutionary algorithms used for the full ultimatum game to the mini ultimatum game? In particular can we test the conclusion in [21] by looking at the time evolution of a population structured on a graph?

Consider the evolutionary dynamics exactly as before with these modifications:
• Selection - Ohtsuki & Nowak introduced a parameter for varying the intensity of selection. If P is the sum of an individuals payoffs, the fitness of an individual is given by $F = 1 - w + wP$ where w is the fitness intensity parameter. $w = 1$ is strong selection and $w \to 0$ is the limit of weak selection. This is similar to the basic fitness of the population C in the derivation of the replicator equation and is realistic as the fitness of an individual depends on many factors and not just a single game interaction. Weak selection ($w \ll 1$) was assumed in their derivations.
• Mutation - Offspring have a small probability, μ, of mutating and when mutating an equal chance of mutating to either of the two other strategies.

We will apply these dynamics to the well-mixed, one-dimensional ring, two-dimensional lattice and random regular graph. In each of the cases below we have used the following

parameters:
• $h = 0.49, l = 0.01$
• The selection intensity, w, is 0.01 i.e. weak selection as in [21].
• The mutation probability, μ, is 0.001 i.e. we can expect, on average, one mutation every ten generations.

3.5.1 Well-mixed

In the well mixed environment we see the evolution of strategies exactly as per the replicator equation. Figure 3.4 shows an example.

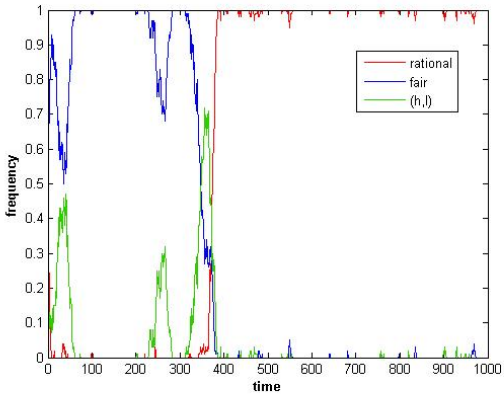


Figure 3.4: Figure shows the time evolution of mini ultimatum game strategies in a well-mixed environment of 100 individuals. The initial strategies (artificially chosen to highlight the dynamics) are in the region of the simplex closer to S_2 (fair) than S_1 (rational) where the solutions moved away from S_1 towards the simplex border between S_2 and S_3. As per the replicator dynamics the frequency of strategy S_2 initially increases, there follows a period of drift between S_2 and S_3 (corresponding to the simplex boundary) as each strategy does as well in a population of the other. Once a critical mass of S_3 exist the S_1 strategists dominate and the frequencies evolve to mostly rational which is uninvadeable.

3.5.2 Random regular graph

The dynamics of the game on the random regular graph do appear to be as predicted by the Ohtsuki & Nowak equations. Figure 3.5 shows an example.

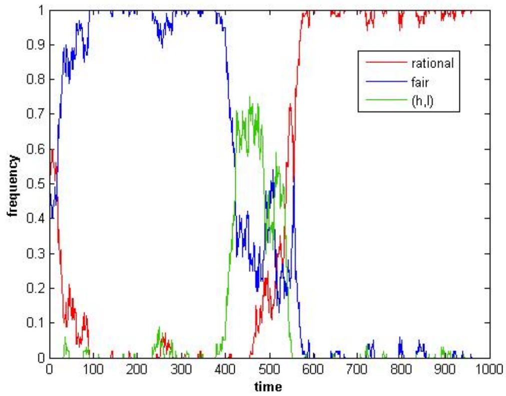


Figure 3.5: Figure shows the time evolution of mini ultimatum game strategies in a population of 100 individuals on a k=4 random regular graph. The initial strategies are the same as the well-mixed above and again chosen to highlight the dynamics. As predicted by the modified replicator equation initially strategy S_2 (fair) fairs better than S_1 (rational) but after a tipping point S_1 dominates and cannot be invaded.

3.5.3 One-dimensional ring

In the full ultimatum game we observed that fair strategies are able to evolve when neighbourhoods are small. We'll see in this section that the evolutionary dynamics for the mini game do not result in the 'fair' strategy dominating. Instead the dynamics allow oscillations in frequency between all three strategies.

For the discussion on one-dimensional rings it will be useful to adopt the following shorthand names for the three strategies:
red - 'rational'
blue - 'fair'
green - (h, l)
and we will keep this colour scheme throughout this section.

It is useful again to consider patterns of the important conflicts that govern the evolution. Here we will just consider the two cases shown in 3.6.

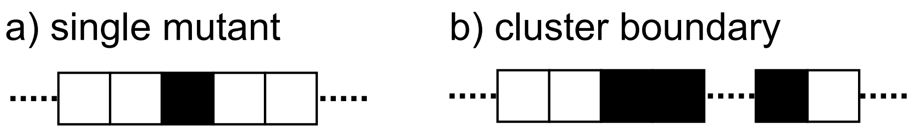


Figure 3.6: mutants and clusters

Some probability calculations show on the above show why we see the oscillations in strategy frequencies. In any one generation, for:
a) single mutant invader - Except for green invading red all mutants have a better than even chance of survival but only red invading green has a better than even chance of spreading.
b) boundary of two clusters - Blue has a slight edge on red, blue and green are evenly

matched and red is stronger than green.

Figure 3.7 shows the time evolution of frequencies and figure 3.8 shows snapshots at certain time intervals for the $k = 2$ case in figure 3.7 which gives some insight into the dynamics.

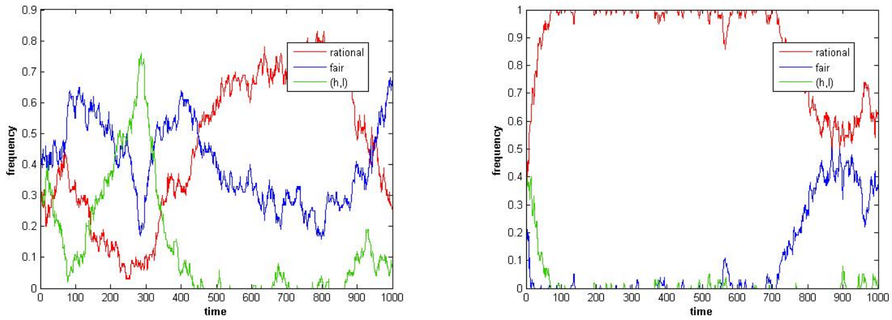


Figure 3.7: Evolution of frequencies in a one-dimensional ring environment for 2 and 4 neighbours. Initial strategies are randomly distributed. Frequencies in the $k = 2$ case are able to oscillate with no strategy dominating the others i.e. all three strategies are invadeable. in the $k = 4$ case we see that red is more dominant but blue is still able to invade. This will become less likely as the neighbourhood size increases.

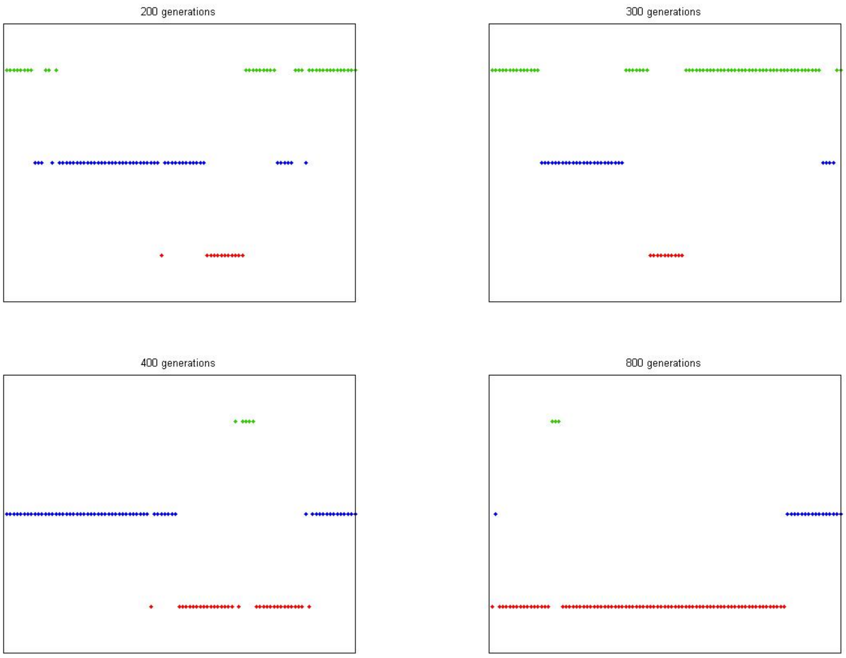


Figure 3.8: Snapshots at various intervals of the evolving population of the $k = 2$ case of figure 3.7. At 200 generations green is growing mostly by competing at cluster boundaries with blue. At 300 generations a cluster of red has formed inside green and would be expected to grow. At 400 generations the large green population has been nearly wiped out in conflicts with both blue and red. There are also two significant blue-red fronts which will evolve as a simple Markov process - red initially growing (near it's maximum at 800 generations), then, by generation 1000, blue fighting back and dominating.

3.5.4 Two-dimensional lattice

In the two-dimensional continuous strategy full ultimatum game we observed that strategies above 0.3 were able to evolve by clusters forming and growing. With the three strategy min game, just as the dynamics were different in one-dimension, clusters with strategy other than rational are not able to survive.

Figure 3.9 shows the time evolution of frequencies from random initial conditions. Observe that S_1 ('rational') is dominant and uninvadeable. Figure 3.10 has snapshots taken at certain time intervals of this population and shows that clusters of S_2 ('fair') and S_3 are unable to survive.

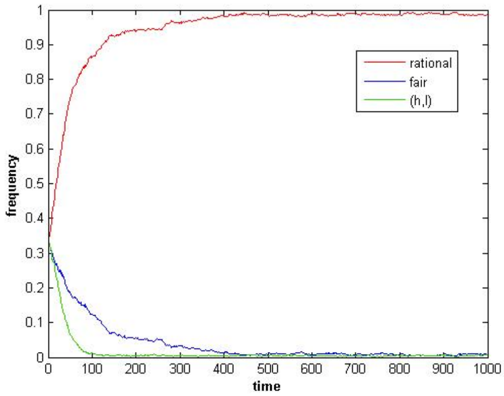


Figure 3.9: Evolution of frequencies in a two dimensional lattice environment. Initial strategies are random. Strategy S_1 dominates and is uninvadeable

DISCUSSION

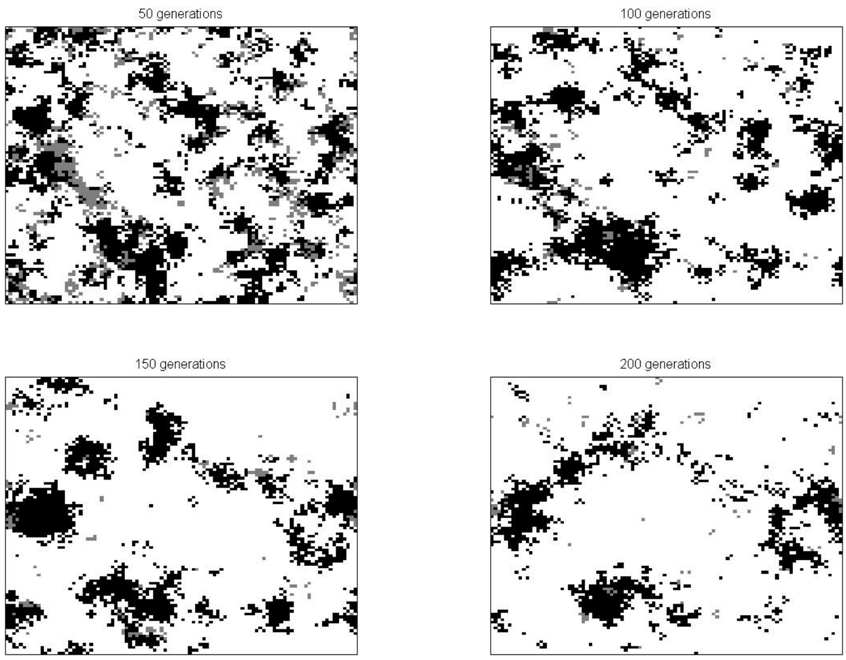


Figure 3.10: Snapshots of the evolution of strategies in a k=4 two dimensional lattice. White cells are strategy S_1 ('rational'), black cells are strategy S_2 ('fair') and grey cells are strategy S_3. Clusters of black and/or grey cells are able to survive invasion by white cells. By 200 generations the greys have all but disappeared and the black cells are being splintered by the white cells. After 400 generations no large black cell clusters will have survived.

4 Spatial replicator equations

4.1 Motivation

One of the observed shortcomings of the standard replicator dynamics is that spatial effects are ignored. "Yet the very concept of evolutionary stability involves the consideration of migrating groups (or invaders or mutants) which produce *ipso facto* spatial variation" [22]. There are a number of ways in which dispersal could be incorporated into equation 3.1 but one of the most widely used is that introduced by Vickers 1989 [22]. Below is a brief outline of that model.

When the dispersal rate is independent of strategy the continuous-time and -space model suggested is

$$\frac{\partial p_r}{\partial t} = p_r[(A\mathbf{p})_r - \mathbf{p}\cdot A\mathbf{p}] + D\nabla^2 p_r, \qquad r = 1,...,m \tag{4.1}$$

where
$p_r = p_r(\mathbf{x}, t)$ is the frequency of strategy r,
$\mathbf{x}$ is the position vector,
D is the dispersal rate,
∇^2 is the Laplacian, and
m is the number of strategies.

This is the heat diffusion

To allow for strategy dependant dispersal rates Vickers introduced number (or number density) and proposed the following extended model:

$$\frac{\partial n_r}{\partial t} = n_r[\frac{(A\mathbf{n})_r}{N} - \frac{\mathbf{n}\cdot A\mathbf{n}}{N^2}] + D_r\nabla^2 n_r, \qquad r = 1,...,m \tag{4.2}$$

where
$n_r = n_r(\mathbf{x}, t)$ is the population density of the r-strategists at position x and time t,
$p_r = \frac{n_r}{N}$ is the frequency of strategy r,
$N = N(\mathbf{x}, t) = \sum_{r=1}^{m} n_r$ is the total number density, and
D_r is the dispersal rate of the r-strategists.

Note that

$$\begin{aligned}
\frac{\partial N}{\partial t} &= \sum_r \frac{\partial n_r}{\partial t} \\
&= \sum_r n_r[\frac{(A\mathbf{n})_r}{N} - \frac{\mathbf{n}\cdot A\mathbf{n}}{N^2}] + \sum_r D_r\nabla^2 n_r \\
&= \sum_r D_r\nabla^2 n_r
\end{aligned}$$

So the total number of individuals can only change by migration across the boundary.

The boundary conditions chosen are usually the zero Neumann condition so that there is no net population flow into or out of the region. Instead in the numerical calculations that follow we will apply the spatial replicator equation to an annulus in the one-dimensional case and a torus in the two-dimensional case so there is no need for boundary conditions.

4.2 Strategy independent dispersal - one space dimension

Applying 4.2 to the mini ultimatum game for the one-dimensional case gives

$$\frac{\partial p_r}{\partial t} = f_r(\mathbf{p}) + D\frac{\partial^2 p_r}{\partial x^2}, \qquad r = 1, 2, 3$$

with $\mathbf{f}_r = p_r[(Ap)_r - \mathbf{p} \cdot A\mathbf{p}]$ i.e.

$$\begin{aligned} f_1 &= p_1[(h-l)p_3 + (h-1)p_2 + p_1p_2] \\ f_2 &= p_2[-hp_1 + p_1p_2] \\ f_3 &= p_3[(l-h)p_1 + p_1p_2] \end{aligned} \tag{4.3}$$

As before we can use $p_3 = 1 - p_1 - p_2$ to reduce this system to the following:

$$\frac{\partial p_r}{\partial t} = f_r(\mathbf{p}) + D\nabla^2 p_r, \qquad r = 1, 2$$

$$\begin{aligned} f_1 &= p_1[(l-1)p_2 + (h-1)(1-p_1) + p_1p_2] \\ f_2 &= p_2[-hp_1 + p_1p_2] \end{aligned} \tag{4.4}$$

Discritising the solution on a two-dimensional grid with space and time indices denoted respectively by i and j we can use the forward time finite difference approximation

$$\frac{\partial p}{\partial t} \approx \frac{p_r^{i,j+1} - p_r^{i,j}}{\Delta t} \tag{4.5}$$

and the central space finite difference approximation

$$\frac{\partial^2 p}{\partial x^2} \approx \frac{p_r^{i+1,j} - 2p_r^{i,j} + p_r^{i-1,j}}{(\Delta x)^2} \tag{4.6}$$

to assemble the following forward marching scheme to solve numerically.

$$p_r^{i,j+1} = p_r^{i,j} + \Delta t f_r^{i,j} + D\frac{\Delta t}{(\Delta x)^2}(p_r^{i+1,j} - 2p_r^{i,j} + p_r^{i-1,j}) \tag{4.7}$$

The scheme is stable for $D\frac{\Delta t}{(\Delta x)^2} < \frac{1}{2}$ so the discritisations must be chosen accordingly.

Figures 4.1 to 4.6 show the results for different dispersal rates. The axes show the discritisation steps rather than dimension units. The axes intervals are $(0, 100)$ for space and $(0, 10)$ time but these are quite arbitrary and are related to the parameter D. High diffusion is analogous to well-mixed and the population evolves rapidly towards the rational. Lower dispersal rates are analogous to a spatial environment in which individuals interact with a neighbourhood. In such an environment wave patterns can emerge. The wave patterns are, however, temporary as no matter the diffusion rate the strategies will ultimately evolve to the rational.

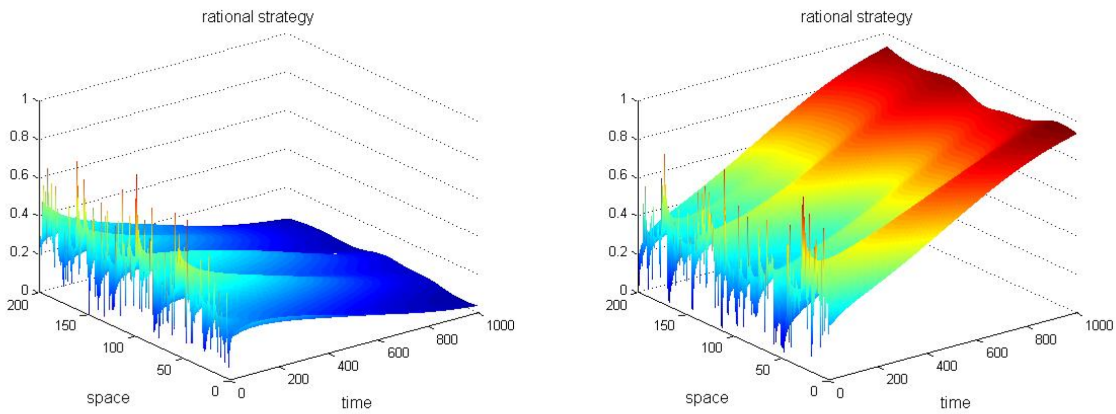


Figure 4.1: High diffusion ($D = 5$). High diffusion is analogous to well-mixed and thus the population evolves quickly to the rational.

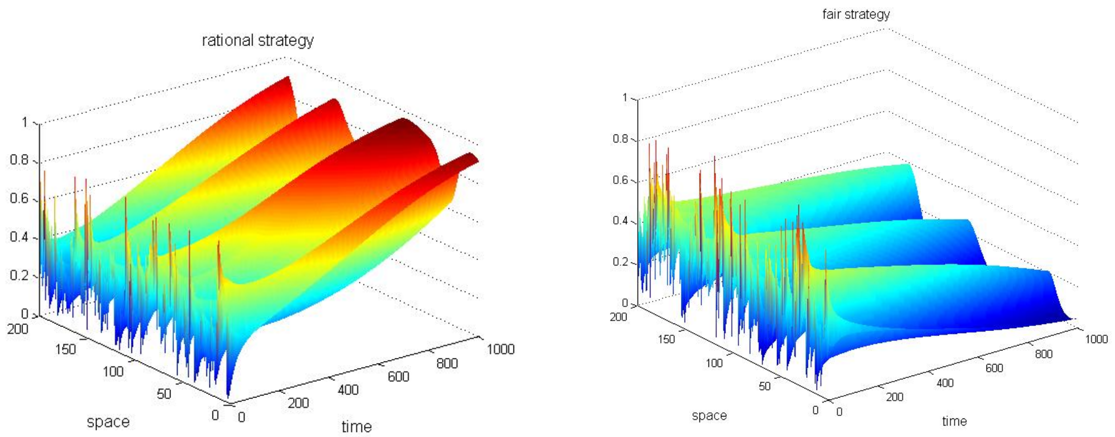


Figure 4.2: Medium diffusion ($D = 1$). Clear wave patterns are beginning to emerge.

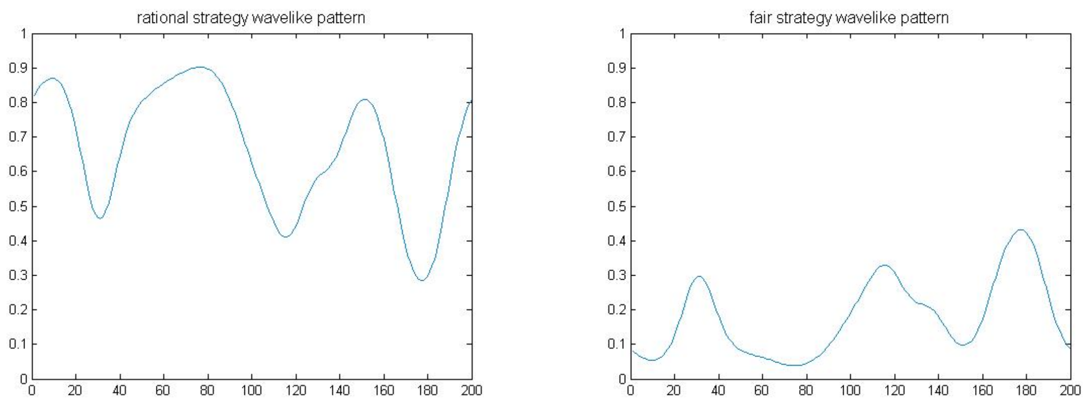


Figure 4.3: Medium diffusion ($D = 1$). The figure is a cross section of figure 4.2 at $t = 8$ or at the $800^t h$ discritisation step and shows a clear wave pattern.

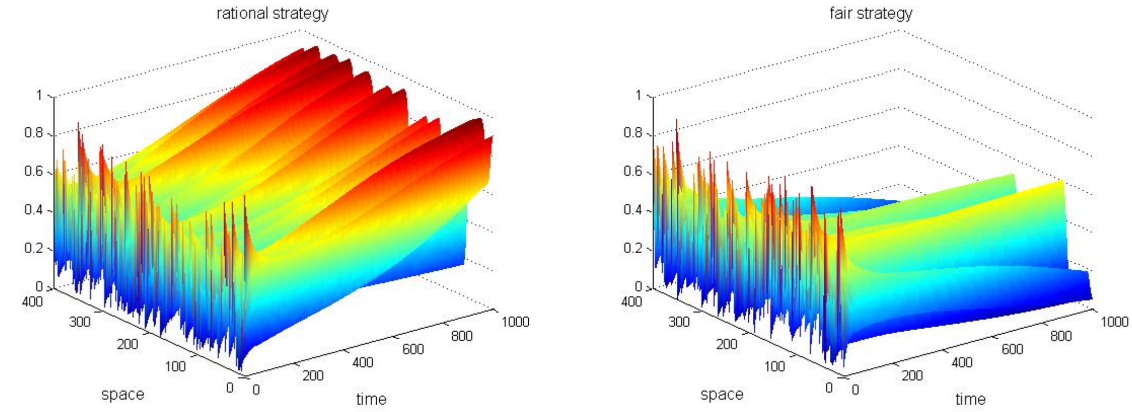


Figure 4.4: Low diffusion ($D = 0.1$). Low diffusion is analogous with interactions in a small neighbourhood. The population evolves slower towards rational and clear wave patterns have emerged

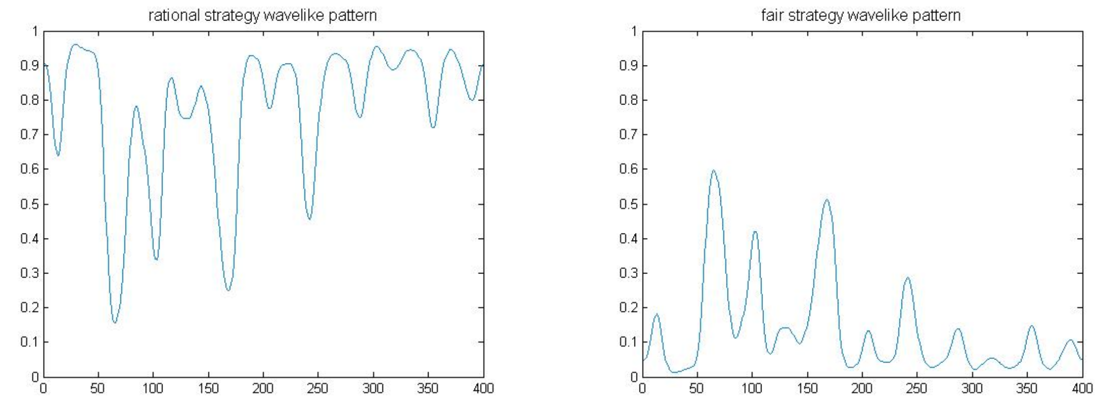


Figure 4.5: Low diffusion ($D = 0.1$). The figure is a cross section at $t = 10$ i.e. at the end of fig 4.4

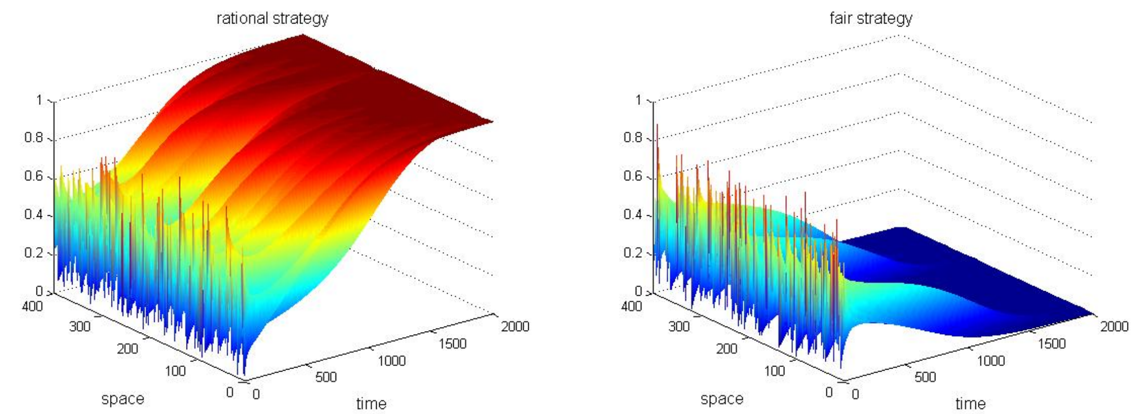


Figure 4.6: The figure shows the low diffusion case ($D = 0.1$) extended to $t = 20$ to demonstrate that solutions still converge to the rational

Diffusion of a cluster

What happens if we begin with a single strong cluster of 'rationals' in a population mad mostly of 'fair' players?

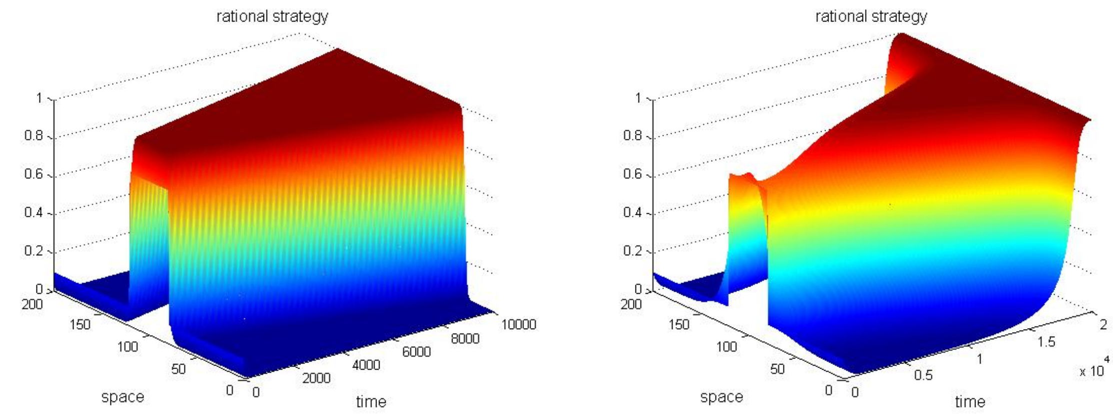


Figure 4.7: The figures show that with either high or low diffusion the strategies will ultimately converge to the rational. The left figure used $D = 10$ and $t = 150$. The right figure used $D = 0.1$ and $t = 400$. The lower D requires many more discritisation steps for stability.

4.3 Strategy independent dispersal in two space dimensions

Adapting the one-dimensional case to two dimensions gives the following system:

$$\frac{\partial p_r}{\partial t} = f_r(\mathbf{p}) + D(\frac{\partial^2 p_r}{\partial x^2} + \frac{\partial^2 p_r}{\partial y^2}), \qquad r = 1, 2 \tag{4.8}$$

with f_r as before.

We now discritise the solution on a three-dimensional lattice denoting the x-space, y-space and time indices respectively by i, j and k. Using the same finite difference approximation for time i.e.

$$\frac{\partial p}{\partial t} \approx \frac{p_r^{i,j,k+1} - p_r^{i,j,k}}{\Delta t} \tag{4.9}$$

and the two-dimensional central space finite difference approximation i.e.

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})p \approx \frac{p_r^{i+1,j,k} - 2p_r^{i,j,k} + p_r^{i-1,j,k}}{(\Delta x)^2} + \frac{p_r^{i,j+1,k} - 2p_r^{i,j,k} + p_r^{i,j-1,k}}{(\Delta y)^2} \tag{4.10}$$

we assemble the following forward marching scheme to solve numerically. Δs denotes the distance between mesh points for both the x and y axis having chosen them to be identical.

$$p_r^{i,j,k+1} = p_r^{i,j,k} + \Delta t f_r^{i,j,k} + D\frac{\Delta t}{(\Delta s)^2}(p_r^{i+1,j,k} + p_r^{i-1,j,k} - 4p_r^{i,j,k} + p_r^{i,j+1,k} + p_r^{i,j-1,k}) \tag{4.11}$$

Figures 4.8 and 4.9 show the results for different dispersal rates. Again, as in one-dimensional space the solutions converge to the rational but wave patterns emerge as the population evolves.

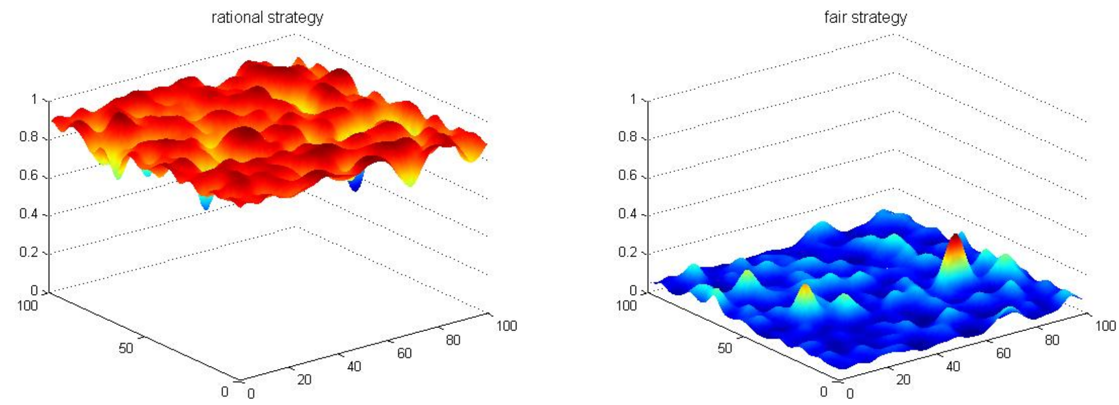


Figure 4.8: High diffusion. Figure shows the the frequency of strategies at the final time step for $D = 0.1$

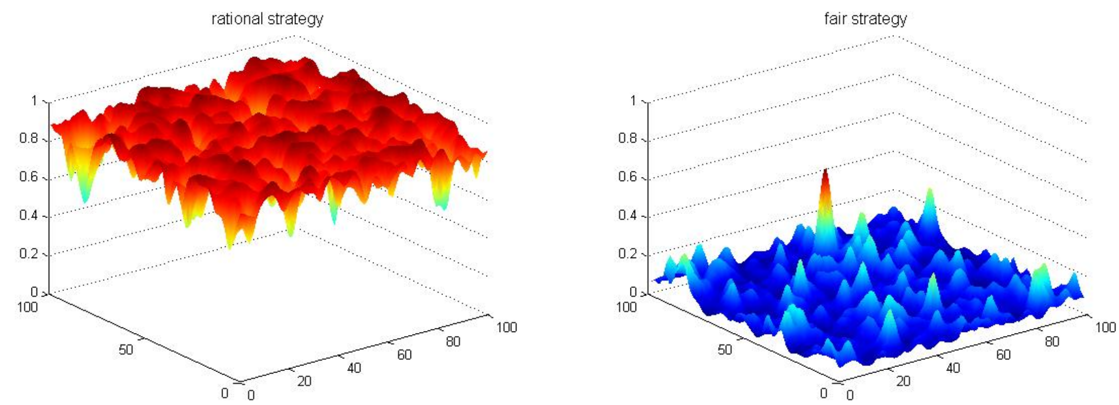


Figure 4.9: Lower diffusion. Figure shows the frequency of strategies at the final time step for $D = 0.05$

4.4 Strategy dependant dispersal

There seems to be no valid reason why any of the three strategies would different dispersal rates, however, if we apply different dispersal rates it does seem that more stable waves can form. Figure 4.10 shows an example. The granularity of the charts is not very good but clearly small clusters of the 'fair' strategy are stable.

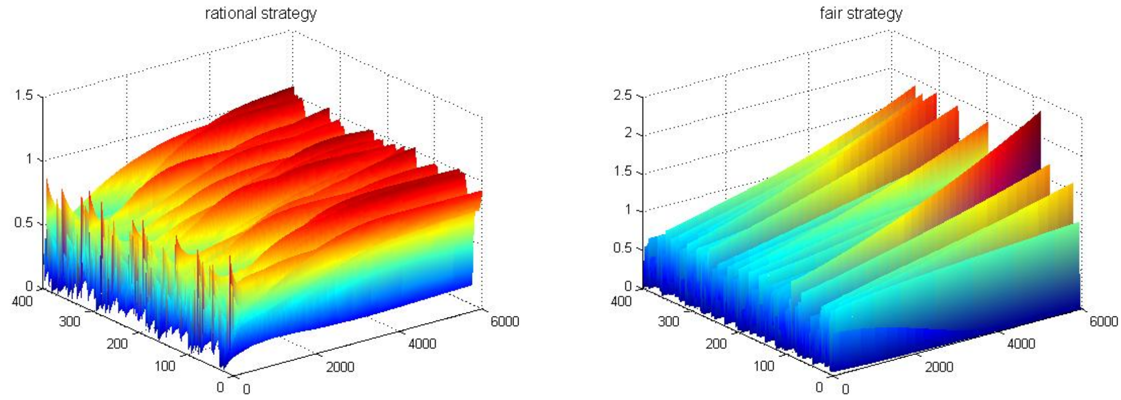


Figure 4.10: Figure shows potentially stable wave patterns emerging with different dispersal rates - $D_2 \ll D_1$

5 Summary and further areas of interest

An evolutionary approach to the ultimatum game in Ch2 demonstrated that, as long as mutations are small, the strategies of a population who encounter each other randomly will tend to close to zero (sec 2.1) i.e. individuals ultimately display 'rational' behaviour.

When the population has some structure and the concept of neighbourhood is introduced 'fair' behaviour only evolves when the neighbourhood is closely knit enough for kin selection to have an impact. We observed in (sec 2.4) that where the neighbourhood was arranged randomly on a graph the strategies tend towards the rational quickly with increasing neighbourhood size, however in (sec 2.2) and (sec. 2.3) we observed that with the neighbourhood arranged spatially and tightly knit fairer strategies can evolve. This may be mirrored by the way human populations for much of their existence have lived in small communities where an individual with whom your neighbour interacts is likely to interact with you.

We also looked at whether fairer strategies could evolve when individuals could label others as 'unfair' and not interact with them. Whilst parameters were kept to reasonable levels players ultimately tended towards the 'rational'. The notions of labels, reputations and memory could be explored in much greater detail.

In Ch3 we applied one of the fundamental tools of evolutionary game theory (the replicator equation) to the mini ultimatum game in a well-mixed environment and on graphs. These were compared to, and showed consistency with, the evolutionary algorithms in Ch2. We observed that it is more difficult for 'fair' strategies to evolve in a spatial context in the mini game than the full game.

Finally in Ch4 we studied the spatial replicator equation applied to the mini ultimatum game in one- and two-dimensional space. We observed that wave patterns can emerge between fair and rational but that ultimately the population diffuses to the rational.

The replicator equation is the standard framework used to study the evolutionary dynamics of a game with a fixed number of strategies. However we have seen that the evolutionary dynamics of the mini game are not quite the same as the full game so it would be useful to use a continuous strategy model. An adaptive dynamics framework has been introduced by Nowak & Sigmund [24] which allows for continuous strategy games but assumes that all individuals in the population play the same strategy. If a mutant can invade the whole population adopts the mutant's strategy. This framework has been applied to the ultimatum game by Page & Nowak [25] where interestingly it was shown that fairness evolves if a small fraction of players offer their acceptance level.

A Stability analysis of the replicator dynamics

A.1 Standard replicator dynamics

The Jacobian of the 2D ODE system is

$$J = \begin{pmatrix} (l-1)x_2 + (h-l)(1-2x_1) + 2x_1x_2 & (l-1)x_1 + x_1^2 \\ -hx_2 + x_2^2 & -hx_1 + 2x_1x_2 \end{pmatrix}$$

Applying the Jacobian at each of the steady states we see the following:
In the below T stands for trace and D for determinant.

$$J_{(1,0)} = \begin{pmatrix} -2(h-l) & l \\ 0 & -h \end{pmatrix}$$

$T = -3h + 2l < 0$
$D = 2h^2 - 2hl > 0$
$\Rightarrow$ stable

$$J_{(0,1)} = \begin{pmatrix} h-1 & 0 \\ 1-h & 0 \end{pmatrix}$$

$T = h - 1 < 0$
$D = 0$
$\Rightarrow$ stable non-isolated fixed point

$$J_{(0,0)} = \begin{pmatrix} h-l & 0 \\ 0 & 0 \end{pmatrix}$$

$T = h - l > 0$
$D = 0$
$\Rightarrow$ unstable

$$J_{(0,x_2)} = \begin{pmatrix} (l-1)x_2 + h - l & 0 \\ -hx_2 + x_2^2 & 0 \end{pmatrix}$$

$T = (l-1)x_2 + h - l$
$D = 0$
$\Rightarrow$ stable for $x_2 > \frac{h-l}{1-l}$

$$J_{(1-h,h)} = \begin{pmatrix} l(1-h) & h(h-l-1)+l \\ 0 & h(1-h) \end{pmatrix}$$

$T = (h+l)(1-h) > 0$
$D = lh(1-h)^2$
$\Rightarrow$ unstable

A.2 Replicator dynamics on a k=3 imitation graph

The Jacobian of the 2D ODE system is

$$J = \begin{pmatrix} (2l - \frac{3}{2})x_2 + 2(h-l)(1-2x_1) + 2x_1x_2 & (2l - \frac{3}{2})x_1 + x_1^2 \\ (\frac{1}{2} - 2h)x_2 + x_2^2 & (\frac{1}{2} - 2h)x_1 + 2x_1x_2 \end{pmatrix}$$

Applying the Jacobian at each of the steady states we see the following:
As before T stands for trace and D for determinant.

$$J_{(1,0)} = \begin{pmatrix} 2(l-h) & 2l - \frac{1}{2} \\ 0 & \frac{1}{2} - 2h \end{pmatrix}$$

$T = \frac{1}{2} + 2l - 4h < 0$
$D = 2(l-h)(\frac{1}{2} - 2h) > 0$
$\Rightarrow$ stable node

$$J_{(0,1)} = \begin{pmatrix} 2h - \frac{3}{2} & 0 \\ \frac{3}{2} - 2h & 0 \end{pmatrix}$$

$T = 2h - \frac{3}{2} < 0$
$D = 0$
$\Rightarrow$ stable non-isolated fixed point

$$J_{(0,0)} = \begin{pmatrix} 2(h-l) & 0 \\ 0 & 0 \end{pmatrix}$$

$T = 2(h-l) > 0$
$D = 0$
$\Rightarrow$ unstable

$$J_{(0,x_2)} = \begin{pmatrix} (2l - \frac{3}{2})x_2 + 2(h-l) & 0 \\ (\frac{1}{2} - 2h)x_2 + x_2^2 & 0 \end{pmatrix}$$

$T = (2l - \frac{3}{2})x_2 + 2(h-l)$
$D = 0$
$\Rightarrow$ stable for $x_2 > \frac{2(h-l)}{\frac{3}{2}-2l}$

$$J_{(\frac{3}{2}-2h, 2h-\frac{1}{2})} = \begin{pmatrix} -4lh + 3l + h - \frac{3}{4} & h(4h-4l-3) + 3l \\ 0 & h(4-4h) - \frac{3}{4} \end{pmatrix}$$

.

$D < 0$
$\Rightarrow$ saddle

A.3 Replicator dynamics on a k=4 imitation graph

The Jacobian of the 2D ODE system is

$$J = \begin{pmatrix} (\frac{10}{7}l - \frac{17}{14})x_2 + \frac{10}{7}(h-l)(1-2x_1) + 2x_1x_2 & (2l - \frac{3}{2})x_1 + x_1^2 \\ (\frac{3}{14} - \frac{10}{7}h)x_2 + x_2^2 & (\frac{3}{14} - \frac{10}{7}h)x_1 + 2x_1x_2 \end{pmatrix}$$

Applying the Jacobian at each of the steady states we see the following:
As before T stands for trace and D for determinant.

$$J_{(1,0)} = \begin{pmatrix} \frac{10}{7}(l-h) & \frac{10}{7}l - \frac{3}{14} \\ 0 & \frac{3}{14} - \frac{10}{7}h \end{pmatrix}$$

$T = \frac{3}{14} + \frac{10}{7}l - \frac{20}{7}h < 0$
$D = \frac{10}{7}(l-h)(\frac{3}{14} - \frac{10}{7}h) > 0$
$\Rightarrow$ stable node

$$J_{(0,1)} = \begin{pmatrix} \frac{10}{7}h - \frac{17}{14} & 0 \\ \frac{17}{14} - \frac{10}{7}h & 0 \end{pmatrix}$$

$T = \frac{10}{7}h - \frac{17}{14} < 0$
$D = 0$
$\Rightarrow$ stable non-isolated fixed point

$$J_{(0,0)} = \begin{pmatrix} \frac{10}{7}(h-l) & 0 \\ 0 & 0 \end{pmatrix}$$

$T = \frac{10}{7}(h-l) > 0$
$D = 0$
$\Rightarrow$ unstable

$$J_{(0,x_2)} = \begin{pmatrix} (\frac{10}{7}l - \frac{17}{14})x_2 + \frac{10}{7}(h-l) & 0 \\ (\frac{3}{14} - \frac{10}{7}h)x_2 + x_2^2 & 0 \end{pmatrix}$$

$T = (\frac{10}{7}l - \frac{17}{14})x_2 + \frac{10}{7}(h-l)$
$D = 0$
$\Rightarrow$ stable for $x_2 > \frac{\frac{10}{7}(h-l)}{\frac{17}{14}-\frac{10}{7}l}$

$$J_{(\frac{17}{14}-\frac{10}{7}h, \frac{10}{7}h-\frac{3}{14})} = \begin{pmatrix} -\frac{100}{49}lh + \frac{85}{49}l + \frac{15}{49}h - \frac{51}{196} & h(\frac{100}{49}h - \frac{100}{49}l - \frac{85}{49}) + \frac{85}{49}l \\ 0 & h(\frac{100}{49} - \frac{100}{49}h) - \frac{51}{96} \end{pmatrix}$$

.

$D < 0$
$\Rightarrow$ saddle

B Samples of code used to generate results

B.1 One-dimensional spatial ultimatum game

```
clear all; clc
tic

% Parameters
n=100;      % players
nbrs=8;     % neighbours
m=10^4;     % generations
w=1;        % selection strength
x=0.001;    % mutation factor
output='chart';
sample_start=10^4;
sample_int=10^3;

% Initial strategies
P=rand(1,n);      % poposer
Q=rand(1,n);      % responder

% Pairings
A=sparse(n,n);
for k=1:nbrs/2
    A=sparse(A+diag(ones(1,length(diag(A,k))),k)...
      +diag(ones(1,length(diag(A,n-k))),n-k));
end
A=A+A';

% Play game through m generations
for k=1:m
    clc; k
    % INITIALISE FITNESS VECTOR
    F=zeros(1,n);

    % GAME
    for i=1:n
        for j=1:n
            Ftmp=zeros(1,n);
            if A(i,j)==1
                Ftmp(i)=(floor(P(i)-Q(j))+1)*(1-P(i)); % i is prop.
                Ftmp(j)=(floor(P(i)-Q(j))+1)*P(i);     % j is resp.
            end
            F=F+Ftmp;
        end
    end

    % SELECTION
   for i=1:n
        Atmp=A(i,:);
        Atmp(i)=1;
        Fi=Atmp.*F;
        if sum(Fi)==0; Fi=Atmp.*(1/(nbrs+1)); end
        Fi=1-w+w*Fi;
        prob=Fi/sum(Fi);
        odds=cumsum(prob);
        selection=find(full(odds)>=rand,1);
        Ps(i)=P(selection);
        Qs(i)=Q(selection);
    end

    % OUTPUT
    switch output
```

```
        case 'chart'
            OutP(k)=mean(Ps);
            OutQ(k)=mean(Qs);
            StdP(k)=std(Ps);
            StdQ(k)=std(Qs);
        case 'table'
            if k==sample_start
                OutP(1)=mean(Ps);
                OutQ(1)=mean(Qs);
            end
            if (k>sample_start)&&(mod(k,sample_int)==0)
                OutP(end+1)=mean(Ps);
                OutQ(end+1)=mean(Qs);
            end
    end

    % MUTATION
    mutateP=(x/2)*(1-2*rand(1,n));
    mutateQ=(x/2)*(1-2*rand(1,n));
    Pm=min(max(Ps+mutateP,0),1);
    Qm=min(max(Qs+mutateQ,0),1);

    % NEXT GEN
    P=Pm;
    Q=Qm;

end
toc

% Display
switch output
    case 'chart'
        p=semilogx(OutP); set(p,'Color','red'); ylim([0 1]); hold
        q=semilogx(OutQ); set(q,'Color','blue');
        l=legend('mean offer','mean response');
        set(l,'Position',[0.56 0.78 0.3 0.1])
        figure
        p=semilogx(StdP); set(p,'Color','red'); ylim([0 1]); hold
        q=semilogx(StdQ); set(q,'Color','blue');
        l=legend('offer stdev','response stdev');
        set(l,'Position',[0.56 0.78 0.3 0.1])
    case 'table'
        pbar=mean(OutP); display(pbar)
        qbar=mean(OutQ); display(qbar)
        pstd=std(OutP); display(pstd)
        qstd=std(OutQ); display(qstd)
end
```

B.2 Two-dimensional spatial ultimatum game

```
clear all; clc
tic

% Parameters
n=100;     % gridsize
m=10^3;    % # of generations
w=1;       % selection strength
x=0.001;   % mutation factor
output='chart';
sample_start=10^4;
sample_int=10^3;

% Initial strategies
P1=rand(n);      % poposer
Q1=rand(n);      % responder

% Play game through m generations
for k=1:m
```

```
% GAME

% Proposer strategies of neighbours
P2=P1(:,[n 1:n-1]);         % left
P3=P1(:,[2:n 1]);           % right
P4=P1([n 1:n-1],:);         % above
P5=P1([2:n 1],:);           % below

% Responder strategies of neighbours
Q2=Q1(:,[n 1:n-1]);         % left
Q3=Q1(:,[2:n 1]);           % right
Q4=Q1([n 1:n-1],:);         % above
Q5=Q1([2:n 1],:);           % below

% Fitness
F1 =    (floor(P1-Q2)+1).*(1.-P1)...
      + (floor(P1-Q3)+1).*(1.-P1)...
      + (floor(P1-Q4)+1).*(1.-P1)...
      + (floor(P1-Q5)+1).*(1.-P1)...
      + (floor(P2-Q1)+1).*P2...
      + (floor(P3-Q1)+1).*P3...
      + (floor(P4-Q1)+1).*P4...
      + (floor(P5-Q1)+1).*P5;           % middle
F2=F1(:,[n 1:n-1]);                     % left
F3=F1(:,[2:n 1]);                       % right
F4=F1([n 1:n-1],:);                     % up
F5=F1([2:n 1],:);                       % down

% SELECTION
F=[F1(1:n^2);F2(1:n^2);F3(1:n^2);F4(1:n^2);F5(1:n^2)]';
F=1-w+w*F;
prob=F./kron(sum(F,2),ones(1,5));
prob(isnan(prob))=1/5;
odds=cumsum(prob,2);
random=kron(rand(n^2,1),ones(1,5));
[c,r]=find((odds-random)'>=0);
A1=[r c];
[idx1,idx2]=unique(A1(:,1),'first');
A2=A1(idx2,:);
selection=reshape(A2(:,2),n,n);
Ps =    (selection==1).*P1 + (selection==2).*P2...
      + (selection==3).*P3 + (selection==4).*P4...
      + (selection==5).*P5;
Qs =    (selection==1).*Q1 + (selection==2).*Q2...
      + (selection==3).*Q3 + (selection==4).*Q4...
      + (selection==5).*Q5;

% OUTPUT
switch output
    case 'chart'
        caxis([0 1])
        a=mean2(Ps);
        b=mean2(Qs);
        subplot(1,2,1); image(Ps*64)
        axis off square
        title({[num2str(k),' generations'];['mean offer = ', num2str(a,3)]})
        subplot(1,2,2); image(Qs*64)
        axis off square
        title(['mean response = ', num2str(b,3)])
        pause(0.001)
    case 'table'
        if k==sample_start
            OutP(1)=mean2(Ps);
            OutQ(1)=mean2(Qs);
        end
        if (k>sample_start)&&(mod(k,sample_int)==0)
            OutP(end+1)=mean2(Ps);
            OutQ(end+1)=mean2(Qs);
        end
end
```

```
    % MUTATION
    mutateP=(x/2)*(1-2*rand(n));
    mutateQ=(x/2)*(1-2*rand(n));
    Pm=min(max(Ps+mutateP,0),1);
    Qm=min(max(Qs+mutateQ,0),1);

    % Next Generation
    P1=Pm;
    Q1=Qm;

end
toc

% Display
switch output
    case 'chart'
        fprintf('mean proposer strategy = %f\n',a)
        fprintf('mean proposer strategy = %f\n',b)
    case 'table'
        pbar=mean(OutP); display(pbar)
        qbar=mean(OutQ); display(qbar)
        pstd=std(OutP); display(pstd)
        qstd=std(OutQ); display(qstd)
end
```

B.3 Cost and refusal

```
clear all; clc
tic

% PARAMETERS
% Population
n=100;        % players
m=100*n;       % interactions per generation
g=10^4;       % generations
% Game Dynamics
r=0.5;            % blackball probability
cost=0.2;    % cost of interaction
% Selection
conf=0.9;    % confidence
% Mutation
x=0.005;       % mutation factor
% Output
output='chart';
sample_start=10^4;
sample_int=10^3;

% Selection scaling
Range=binoinv(conf,m,2/n);
xmin=Range*(-cost);
xmax=Range;
ymin=0;
ymax=Range;

% Initial strategies
P=rand(1,n);       % proposer
Q=rand(1,n);       % responder

% Play game through g generations
for k=1:g

    % INITIALISE FITNESS VECTOR AND BLACKBALL MATRIX
    F=zeros(1,n);
    B=zeros(n);

    % GAME through m interactions
    for i=1:m
```

```
        % Randomly choose players
        a=ceil(n*rand);          % proposer
        b=ceil(n*rand);          % responder
        while b==a; b=ceil(n*rand); end    % can't play yourself

        % Will they play
        % Only possible to score if not blackballed
        if B(a,b)==1
             game=0;
        else game=1;
        end

        % Payoffs
        F(a)=F(a)+game*floor((P(a)-Q(b))+1)*(1-P(a)-cost);
        F(b)=F(b)+game*floor((P(a)-Q(b))+1)*(P(a)-cost);

        % Blackball with probability b
        % If previously blackballed - remain blackballed
        if game==1
            B(a,b)=floor(P(a)-Q(b))*-1*(ceil(r-rand));
            B(b,a)=B(a,b);
        end
    end

    % SELECTION
    F=(ymax-ymin)/(xmax-xmin)*(F-xmin)+ymin;       % linear scaling
    F(F<0)=0;
    prob=F/sum(F);
    prob(isnan(prob))=1/n;
    odds=cumsum(prob);
    random=rand(1,n);
    for i=1:n
        selection(i)=find(odds>=random(i),1);
        Ps(i)=P(selection(i));
        Qs(i)=Q(selection(i));
    end

    % OUTPUT
    switch output
        case 'chart'
            OutP(k)=mean(Ps);
            OutQ(k)=mean(Qs);
            StdP(k)=std(Ps);
            StdQ(k)=std(Qs);
        case 'table'
            if k==sample_start
                OutP(1)=mean(Ps);
                OutQ(1)=mean(Qs);
            end
            if (k>sample_start)&&(mod(k,sample_int)==0)
                OutP(end+1)=mean(Ps);
                OutQ(end+1)=mean(Qs);
            end
    end

    % MUTATION
    mutateP=(x/2)*(1-2*rand(1,n));
    mutateQ=(x/2)*(1-2*rand(1,n));
    Pm=min(max(Ps+mutateP,0),1);
    Qm=min(max(Qs+mutateQ,0),1);

    % NEXT GEN
    P=Pm;
    Q=Qm;

end
toc

% Display
switch output
```

```
    case 'chart'
        figure1 = figure('Color',[1 1 1]);
        p=semilogx(OutP); set(p,'Color','red'); ylim([0 1]); hold
        q=semilogx(OutQ); set(q,'Color','blue');
        l=legend('mean offer','mean response');
        set(l,'Position',[0.56 0.78 0.3 0.1])
        set(l,'fontsize',12)
        figure2 = figure('Color',[1 1 1]);
        p=semilogx(StdP); set(p,'Color','red'); ylim([0 1]); hold
        q=semilogx(StdQ); set(q,'Color','blue');
        l=legend('offer stdev','response stdev');
        set(l,'Position',[0.56 0.78 0.3 0.1])
        set(l,'fontsize',12)
    case 'table'
        pbar=mean(OutP); display(pbar)
        qbar=mean(OutQ); display(qbar)
        pstd=std(OutP); display(pstd)
        qstd=std(OutQ); display(qstd)
end
```

B.4 Spatial replictor equation in one space dimension

```
clear all; clc
tic

% Parameters and Grid
h=0.49;
l=0.02;
D=0.5;                    % dispersal rate / diffusion constant
nt=1000;                    % # of time steps
nx=400;                     % # of space (x) steps
tmax=10;                   % time end point
xmax=100;                   % space end point

% Grid
dt=tmax/(nt-1);           % time step size
dx=xmax/(nx-1);           % space step size
r=D*dt/(dx)^2; %r2=1-2*r;

% Stability
display(r);

% Create Matrices with initial conditions
P1=zeros(nx,nt); P2=zeros(nx,nt);
Pinit=zeros(1,3); count=1;
while size(Pinit,1)<nx
    p1=rand; p2=rand;
    if p1+p2<=1;
        Pinit(count,1)=p1;
        Pinit(count,2)=p2;
        Pinit(count,3)=1-p1-p2;
        count=count+1;
    end
end
P1(:,1)=Pinit(:,1); P2(:,1)=Pinit(:,2);
%[Pinit sum(Pinit,2)]

% Forward time marching - No boundary conditions as on a ring
for j=2:nt   % time index
    for i=1:nx   % space (x) index

        f1=P1(i,j-1)*((l-1)*P2(i,j-1)+(h-l)*(1-P1(i,j-1))...
            +P1(i,j-1)*P2(i,j-1));
        f2=P2(i,j-1)*(-h*P1(i,j-1)+P1(i,j-1)*P2(i,j-1));

        if i==1
          P1(i,j)=...
            P1(i,j-1)+dt*f1 ...
```

```
          +r*(P1(i+1,j-1)-2*P1(i,j-1)+P1(nx,j-1));
        P2(i,j)=...
          P2(i,j-1)+dt*f2 ...
          +r*(P2(i+1,j-1)-2*P2(i,j-1)+P2(nx,j-1));

      elseif i==nx
        P1(i,j)=...
          P1(i,j-1)+dt*f1 ...
          +r*(P1(1,j-1)-2*P1(i,j-1)+P1(i-1,j-1));
        P2(i,j)=...
          P2(i,j-1)+dt*f2 ...
          +r*(P2(1,j-1)-2*P2(i,j-1)+P2(i-1,j-1));

      else
        P1(i,j)=...
          P1(i,j-1)+dt*f1 ...
          +r*(P1(i+1,j-1)-2*P1(i,j-1)+P1(i-1,j-1));
        P2(i,j)=...
          P2(i,j-1)+dt*f2 ...
          +r*(P2(i+1,j-1)-2*P2(i,j-1)+P2(i-1,j-1));
      end

    end     % space(x)
end     % time

%P3=1-P1-P2;
toc

figuresurfr(P1)
figuresurff(P2)
```

Bibliography

[1] Güth, W., Schmittenberger, R. & Schwarze, B. 1982 An experimental analysis of ultimatum bargaining. *J. Econ. Behav. Organ.* **3**, 376-388.

[2] Hofbauer, J. & Sigmund, K., 1998 Evolutionary games and population dynamics. (Cambridge Univ. Press, Cambridge)

[3] Nowak, M.A., Page, K.M. & Sigmund, K. 2000 Fairness Versus Reason in the Ultimatum Game. *Science* Vol 289 1773-1775

[4] Page, K. M., Nowak, M.A., Sigmund, K. 2000 The spatial ultimatum game. *Proc. R. Soc. Lond.* **267**, 2177-2182.

[5] Thaler, R. H. 1988 The ultimatum game. *J. Econ. Behav. Perspect.* **2**, 195-206.

[6] Gth, W. & Tietze, R. 1990 Ultimatum bargaining behaviour, a survey and comparison of experimental results. *J. Econ. Behav. Psychol.* **11**, 417-449.

[7] Roth, A. E., Prasknikar, V., Okuno-Fujiwara, M. & Zamir, S. 1991 Bargaining and market power in Jerusalem, Ljubljana, Pittsburgh and Tokyo. *Am. Econ. Rev.* **81**, 1068-1095.

[8] Bolton, G. E. & Zwick, R. 1995 Anonymity versus punishment in ultimatum bargaining. *Games Econ. Behav.* textbf10, 95-21.

[9] Roth, A.E. 1995 Bargaining Experiments. in *Handbook of experimental economics* (ed. J. Kagel & A.E. Roth), pp. 171-202. Princeton University Press.

[10] Camerer, Colin F. & Thaler, R. H. 1995 Anomalies: ultimatums, dictators and manners. *J. of Econ. Perspect.* **9(2)**, 20919.

[11] Cameron, Lisa A. 1999 Raising the Stakes in the Ultimatum Game: Experimental Evidence From Indonesia. *Econ. Inquiry*, **37(1)** 4759.

[12] Hoffman, E., McCabe, K. A., Shachat, K. & Smith, V. L. 1994 Preferences, Property Rights, and Anonymity in Bargaining Games. *Games and Econ. Behav.* **7(3)** 34680.

[13] Hoffman, E., McCabe, K. A., Shachat, K. & Smith, V. L. 1996 On Expectations and Monetary Stakes in Ultimatum Games. *Intern. J. of Game Theory*, **25(3)** 289301.

[14] Tompkinson, P. & Bethwaite, J. 1995 The Ultimatum Game: Raising the Stakes. *J. of Econ. Behav. and Org.* **27(3)** 43951.

[15] Brosnan, F. S. & De Waal, B. M. F. 2003 Monkeys reject unequal pay. *Nature* **425** 297-9.

[16] Henrich, J. 2000 Does Culture Matter in Economic Behavior? Ultimatum Game Bargaining Among the Machiguenga of the Peruvian Amazon. *The Am. Econ. Rev.* **90(4)** 973-9.

[17] Kuperman, M. N. & Risau Gusman, S. 2008 The effect of the topology on the spatial ultimatum game. *The Eur. Phys. Jn. B* **62** 233-8.

[18] Maslov, S. & Sneppen, K. 2002 Specificity and stability in topology of protein networks. *Science* **296** 910-3.

[19] Maslov, S. Degree-preserving random rewiring and correlation profile of a complex network. <http://www.cmth.bnl.gov/ maslov/matlab.htm>.

[20] Taylor, P.D., Jonker, L. 1978 Evolutionary stable strategues and game dynamics. *Math. Biosci.* **40** 145.

[21] Ohtsuki, H., Nowak, M. 2006 The replicator equation on graphs. *J. of Theor. Biol.* **243** 86-97.

[22] Vickers, G. T. 1989 Spatial patterns and ESS's. *J. of Theor. Biol.* **140** 129-35.

[23] Cressman, R. & Vickers, G. T. 1996 Spatial and density effects in evolutionary game theory. *J. of Theor. Biol.* **184** 359-69.

[24] Nowak, M. A., & Sigmund, K. 1990 The evolution of reactive strategies in iterated gmes. *Acta Applicandandae Math* **20** 247-65.

[25] Page, K. M. & Nowak, M. A. 2000 A generalised adaptive dynamics framework can describe the evolutionary ultimatum game. *J. Theor. Biol.* **209** 173-9.